ROBUST FILTER DESIGN FOR UNCERTAIN STOCHASTIC TIME-DELAY SYSTEMS

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Abstract: The Kalman filter is the optimal linear least-mean-squares-estimator for systems that are described by linear state-space Markov models. In this note, we show how to design robust filters that ensure a minimum bounded error variance for models with mixed stochastic and deterministic uncertainties, as well as with time delays and nonlinearities. We also show how to design filters that simultaneously guarantee an exponential rate of decay and meet a robust performance level.

Keywords: Kalman filter, state-space model, time delay sytem

1. INTRODUCTION

The Kalman filter is the optimal linear least-mean-squares-estimator for systems that are described by linear state-space Markov models.
However, when the model is not accurately known, the performance of the filter can deteriorate appreciably. In this note, we show how to design robust filters that ensure a minimum bounded error variance for models with mixed stochastic and deterministic uncertainties, as well as with time delays and nonlinearities. We also show how to design filters that simultaneously guarantee an exponential rate of decay and meet a robust performance level. Thus, consider the following *n-dimensional* state-space model:

$$x_{k+1} = (A + \Delta A_k)x_k + A_d x_{k-\tau} + Bu_k + Df(x_k) \quad (1)$$

$$y_k = Cx_k + v_k$$
$$z_k = Lx_k, \qquad k \geq 0 \quad (2)$$

where $\{u_k, v_k\}$ are uncorrelated zero-mean random variables with unknown but bounded covariance matrices, say $Eu_k u_k^T < \rho_u I$ and $Ev_k v_k^T < \rho_v I$. The initial state x_0 is also a zero-mean random variable that is uncorrelated with $\{u_k, v_k\}$ for all *k*. The state matrices $\{A, A_d\}$, and the output matrix *C*, are unknown but lie inside a convex polytopic set. That is $(A, A_d, C) \in K$, where *K* is a convex bounded polyhedral domain described by *p* vertices as follows:

$$K = \{(A, A_d, C) = \sum_{i=1}^{i=p} \alpha_i (A_i, A_{di}, C_i),\ \alpha \geq 0,\ \sum_{i=1}^{i=p} \alpha_i = 1\}. \quad (3)$$

Note that although the matrices $\{A, A_k\}$ are constant, the coefficient matrix in *(1)* is time variant due to the presence of the uncertainties ΔA_k. These uncertainties are assumed to be random in nature and are modelled as $\Delta A_k = E\Delta_k G$ where *E* and *G* are known matrices, while Δ_k is a random matrix whose entries have zero mean and are uncorrelated with each other.

The variances of the entries of Δ_k are assumed unknown but bounded by ρ_Δ, $E\Delta_k \Delta_k^T \leq \rho_\Delta I$. The function $f(\cdot)$ in (1) accounts for unmodeled nonlinearities and it is assumed to satisfy $\|f(k)\| \leq \|Ux_k\|$, for some matrix *U*. Observe that model (1) and (2) incorporates both stochastic and deterministic uncertainties (due to the presence of ΔA_k), and deterministic uncertainties (represented by the polyhedral domain *K*). In this note, we investigate the design of linear estimator for $\{x_k, z_k\}$ of the form

$$\hat{x}_{k+1} = A_f \hat{x}_k + B_f y_k \quad \hat{z}_k = L_f \hat{x}_k,\ k \geq 0 \quad (4)$$

where the constant matrices $\{A_f, B_f, L_f\}$ are filters parameters to be determined in order to meet certain performance criteria, including robustness, exponential stability, and bounded state error variance. In *(4)* the notation $\hat{x}_k$ and $\hat{z}_k$ denote the estimates of x_k and z_k, respectively, that are based on $\{y_0, y_1, ..., y_{k-1}\}$. Let $\tilde{x}_k = x_k - \hat{x}_k$, denote the state error vector. It follows from (1) and (4) that the extended state vector $\eta_k = col\{x_k, \hat{x}_k\}$ satisfies

$$\eta_{k+1} = (\bar{A} + \Delta \bar{A}_k)\eta_k + \bar{B}\, w_k + \bar{A}_d\, \eta_{k-\tau} + \bar{D}_f(M\eta_k) \qquad (5)$$

while the output error is given by $\bar{z}_k = z_k - \hat{z}_k = [L - L_f L_f]\eta_k$ and where we are defining the extended quantities

$$\eta_k = \begin{pmatrix} x_k \\ \hat{x}_k \end{pmatrix}, \qquad w_k = \begin{pmatrix} w_k \\ v_k \end{pmatrix}, \qquad \Delta \bar{A}_k = \bar{E}\,\Delta_k \bar{G}$$

$$\bar{E} = \begin{pmatrix} E \\ E \end{pmatrix} \qquad \bar{G} = (G \quad 0) \qquad M = (I \quad 0)$$

$$\bar{D} = \begin{pmatrix} D \\ D \end{pmatrix} \qquad \bar{A}_d = \begin{pmatrix} A_d & 0 \\ A_d & 0 \end{pmatrix} \qquad .$$

$$\bar{A} = \begin{pmatrix} A & 0 \\ A - A_f - B_f C & A_f \end{pmatrix}$$

$$\bar{B} = \begin{pmatrix} B & 0 \\ B & -B_f \end{pmatrix}$$

Definition 1 [Stability With Probability 1]: The stochastic process η_k of *(5)* will be said to be stable with probability 1 if and only if, for any $\delta > 0$ and $\varepsilon > 0$, there exists a $\sigma(\delta, \varepsilon) > 0$ such that if $\|\eta_0\| \le \sigma(\delta, \varepsilon)$, then $P[\sup\|\eta_k\| \ge \varepsilon] \le \delta$. If $P[\sup\|\eta_k\| \ge \varepsilon] \le \delta$ holds for all η_0, then we say that the systems is stable at large.

Definition 2 [Asymptotic Stability]: The stochastic process η_k of *(5)* will be said to be asymptotically stable with probability 1 if and only if it is stable at large and $\|\eta_k\| \to 0$ with probability 1as $k \to \infty$ for any η_0.

Definition 3 [Exponential Stability]: The stochastic process η_k of *(5)* will be said to be exponentially stable with lavel $0 < \varsigma < 1$, if there are real numbers $\mu > 0$, and $\upsilon > 0$ such that $E\|\eta_k\|^2 \le \mu\|\eta_0\|^2 \varsigma^k + \nu$ for any η_0.

Our objective is to determine filter parameters $\{A_f, B_f, L_f\}$ in (4) such that for all admissible uncertainties in the model (1), (2), the augmented (5) is asymptotically stable in the absence of noises and, when noises are present, the state estimation error $\bar{x}_k$ is exponentially stable, independent of the unknown time-delay τ.

2. ASYMPTOTIC STABILITY

Assume that the noise w_k is absent from *(5)* so that

$$\eta_{k+1} = (\bar{A} + \Delta \bar{A}_k)\eta_k + \bar{A}_d\, \eta_{k-\tau} + \bar{D}_f(M\eta_k) \;\; k \ge 0, \qquad (6)$$

Introduce the vector $\phi_k = [\eta_k^T, \eta_{k-1}^T, ..., \eta_{k-r}^T]^T$. We shall seek a Lyapunov Krasovskii *V(.)* of the form $V(\phi_k) = \eta_k^T P \eta_k + \sum_{i=k-r}^{i=k-1} \eta_i^T R \eta_i$, for some positive-definite matrices *P* and *R* to be chosen. Assume, for the moment, that the triplet (A, A_d, C) in (1) and (2) is fixed, i.e, ignore the polytopic set (3).

***Theorem 1 (Asymptotic Stability)*:** Given scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and $0 < \alpha < 1$, if there exist matrices $\{A_f, B_f, P > 0, R > 0\}$, and a scalar $\beta > 0$, such that

$$\begin{pmatrix} \beta I & \bar{A}^T \\ \bar{A} & I \end{pmatrix} > 0 \qquad (7)$$

and

$$W \overset{\Delta}{=} \begin{pmatrix} H & -\bar{A}^T P \bar{A}_d \\ -\bar{A}_d^T P \bar{A} & R - \beta \in_2^{-1} I - \bar{A}_d^T P \bar{A}_d \end{pmatrix} \qquad (8)$$

where

$$H = P - R - \rho_\Delta \bar{G}^T \bar{E}^T P \bar{E} \bar{G} - \bar{A}^T P \bar{A} - \lambda_{\max} (\bar{D}^T P \bar{D}) \bar{U}^T \bar{U} - \beta \varepsilon_1^{-1} I - (\varepsilon_1 + \varepsilon_2)\lambda_{\max}(\bar{D}^T P^2 \bar{D}) \bar{U}^T \bar{U}$$

and $\bar{U} = UM$, then the process $\{\eta_k\}$ of (5), with fixed (A, A_d, C), will be asymptotically stable in the absence of noise for this choice of $\{A_f, B_f\}$.

Proof: Note that

$$\begin{aligned} E[V(\phi_{k+1}) \mid \phi_k, ...\phi_0] - V(\phi_k) &\le \eta_k^T \bar{A} P \bar{A} \eta_k - \eta_k^T P \eta_k \\ &+ \rho_\Delta \eta_k^T \bar{G}^T \bar{E}^T P \bar{E} \bar{G} \eta_k + \eta_k^T \bar{A}^T P \bar{A}_d\, \eta_{k-\tau} \\ &+ \eta_{k-\tau}^T \bar{A}_d^T P \bar{A} \eta_k + \eta_k^T \bar{A}^T P \bar{D} f(M\eta_k) \qquad (9) \\ &+ f^T(M\eta_k) \bar{D}^T P \bar{A} \eta_k + f^T(M\eta_k) \bar{D}^T P \bar{D} f(M\eta_k) \\ &+ f^T(M\eta_k) \bar{D}^T P \bar{A}_d\, \eta_{k-\tau} + \eta_k^T R \eta_k - \eta_{k-\tau}^T R \eta_{k-\tau}. \end{aligned}$$

Now, it is a well-known result [17] that for any real matrices *{X,Y,Z}* with $JJ^T \le \mu I$, it holds for any scalars $\varepsilon > 0$ that

$$XJY + Y^T J^T X^T \le \varepsilon^{-1} \mu X X^T + \varepsilon Y^T Y .$$

From (7), we have $\bar{A}^T \bar{A} < \beta I$. Choosing $J = \bar{A}^T$, we can write

$$\eta_k^T \bar{A}^T P \bar{D} f(M\eta_k) + f^T(M\eta_k) \bar{D}^T P \bar{A} \eta_k \le \beta \in_1^{-1} \eta_k^T \eta_k + \in_1 \lambda_{\max} (\bar{D}^T P^2 \bar{D}) \eta_k^T \bar{U}^T \bar{U} \eta_k .$$

Similarly

$$\eta_{k-\tau}^T \bar{A}_d^T P \bar{D} f(M\eta_k) + f^T(M\eta_k) \bar{D}^T P \bar{A} \eta_{k-\tau} \le \beta \in_2^{-1} \eta_{k-\tau}^T \eta_{k-\tau} + \in_2 \lambda_{\max} (\bar{D}^T P^2 \bar{D}) \eta_k^T \bar{U}^T \bar{U} \eta_k .$$

for some $\varepsilon_1 > 0, \ \ \varepsilon_2 > 0$ and, moreover,

$f^T(M\eta_k)\bar{D}^T P \bar{D} f(M\eta_k) \le \lambda_{\max}(\bar{D}^T P \bar{D})\eta_k^T \bar{U}^T \bar{U}\eta_k$.

Then, we have

$$E[V(\phi_{k+1}) \mid \phi_k, \phi_{k-1}, \ldots \phi_0] - V(\phi_k) \le -\begin{bmatrix} \eta_k^T & \eta_{k-\tau}^T \end{bmatrix} W \begin{bmatrix} \eta_k^T & \eta_{k-\tau}^T \end{bmatrix}^T . \qquad (10)$$

From (8), we get

$$E[V(\phi_{k+1}) \mid \phi_k, \phi_{k-1}, \ldots \phi_0] - V(\phi_k) \le -\alpha \|\eta_k\|^2 < 0$$

which implies that the process $\{\phi_k\}$, and consequently $\{\eta_k\}$, is asymptotically stable. Now, assume that we restrict our choice of P to block diagonal positive-definite matrices, and partition {P,R} in conformity with η_k, and define Q_1 and Q_2, respectively, as

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \ R = \begin{pmatrix} R_1 & R_3 \\ R_3^T & R_2 \end{pmatrix} \ Q_1 = A_f^T P_2 \ \ Q_2 = B_f^T P_2$$

We can see that the requirement (8) is satisfied if

$$\begin{pmatrix} R_1 & R_3 \\ R_3^T & R_2 \end{pmatrix} > 0 \ \begin{pmatrix} \sigma_1 I & D^T P_1 & D^T P_2 \\ P_1 D & P_1 & 0 \\ P_2 D & 0 & P_2 \end{pmatrix} > 0 \qquad (11)$$

$$\begin{pmatrix} \sigma_2 I & D^T P_1 & D^T P_2 \\ P_1 D & I & 0 \\ P_2 D & 0 & I \end{pmatrix} > 0 \qquad (12)$$

$$\begin{pmatrix} S - \alpha I & -R_3 & \bar{J} & 0 & A^T P_1 & \hat{J} & 0 & 0 \\ -R_3^T & P_2 - \beta\varepsilon_1^{-1} I - \alpha I & -Q_1 A_d & 0 & Q_1 & 0 & & \\ \hat{J}^T & -A_d^T Q_1^T & R_1 - \beta\varepsilon_2^{-1} I & R_3 & 0 & 0 & A_d^T P_2 & \\ 0 & 0 & R_3^T & R_2 - \alpha I - \beta \in_2^{-1} I & 0 & 0 & & \\ P_1 A & 0 & 0 & 0 & P_1 & 0 & 0 & 0 \\ \hat{J}^T & Q_1^T & 0 & 0 & P_2 & 0 & 0 & \\ 0 & 0 & P_1 A_d & 0 & 0 & 0 & P_1 & 0 \\ 0 & 0 & P_2 A_d & 0 & 0 & 0 & 0 & P_2 \end{pmatrix} > 0 \qquad (13)$$

where

$$S' = P_1 - \rho_\Delta G^T E^T (P_1 + P_2 EG) - R_1 - (\sigma_1 + (\varepsilon_1 + \varepsilon_2)\sigma_2) U^T U - \beta\varepsilon_1^{-1} I$$

$$\hat{J} \overset{\Delta}{=} -C^T Q_2 - Q_1 + A^T P_2 \qquad (14)$$

$$\hat{J} \overset{\Delta}{=} -A^T(P_1 + P_2)A_d + C^T Q_2 A_d + Q_1 A_d . \qquad (15)$$

The second inequality in (11) guarantees $\lambda_{\max}(\bar{D}^T P \bar{D}) < \sigma_1$, while (12) guarantees $\lambda_{\max}(\bar{D}^T P^2 \bar{D}) < \sigma_2$. These inequalities do not require the $\lambda_{\max}(.)$ operations for *H* and *S* and, therefore, they are linear inequalities in the unknowns.

3. EXPONENTIAL PERFORMANCE

We now show that the process η_k is also exponentially mean square-stable, as well as almost surely stable in norm. To begin with, in the presence of measurement and process noises, and with $\{A_f, B_f\}$ chosen from the feasible solution of (7) and (11)-(13), we obtain the following inequality by repeating the argument of Theorem 1:

$$E[V(\phi_{k+1}) \mid \phi_k, \phi_{k-1}, \ldots \phi_0] - V(\phi_k) \le -\alpha\|\eta_k\|^2 + \rho_u Tr(B^T(P_1 + P_2)B) + \rho_v Tr(B_f^T P_2 B_f) \cdot$$

Now, note that $V(\phi_k) = \phi_k^T \Gamma \phi_k$, where

$$\Gamma \overset{\Delta}{=} diag\{P, R, \ldots\ldots\ldots R\} .$$

It follows that

$$E[V(\phi_{k+1}) \mid \phi_k, \phi_{k-1}, \ldots \phi_0] - V(\phi_k) \le -\psi_k V(\phi_k) + \rho_u Tr(B^T(P_1 + P_2)B) + \rho_v Tr(B_f^T P_2 B_f) \qquad (16)$$

where $\psi_k \overset{\Delta}{=} (\alpha\|\eta_k\|^2 / \lambda_{\max}(\Gamma)\|\phi_k\|^2)$. If *{P,R}* are further chosen such that $\Gamma > I$, then $0 < \psi_k < 1 - \delta$, for some $\delta > 0$.

Consequently

$$E[V(\phi_{k+1}) \mid \phi_k, \phi_{k-1}, \ldots \phi_0] - \frac{V(\phi_k)}{\theta} \le_u Tr(B^T(P_1 + P_2)B) + \rho_v Tr(B_f^T P_2 B_f) \qquad (17)$$

where $\theta = \inf_k (1/1 - \psi_k)$. Let $\psi = \sup_k \psi_k$. Then, $0 < \psi < 1$ and

$$E[V(\phi_{k+1}) \mid \phi_k, \phi_{k-1}, \ldots \phi_0] - V(\phi_k) \le -\psi V(\eta_k) + \rho_u Tr(B^T(P_1 + P_2)B) + \rho_v Tr(B_f^T P_2 B_f) \qquad (18)$$

Inequality (18) allows us to establish that the process $\{\eta_k\}$ is exponentially mean-square stable. In order to arrive at this conclusion, we call upon the following auxiliary results.

Lemma 1: If there exist positive real numbers λ, μ, ν, and $0 < \psi < 1$ such that

$$\mu\|\phi_k\|^2 \le V(\phi_k) \le \nu\|\phi_k\|^2 \qquad (19)$$

and

$$E[V(\phi_{k+1}) \mid \phi_k, \phi_{k-1}, \ldots \phi_0] - V(\phi_k) \le \lambda - \psi V(\phi_k) \qquad (20)$$

then the process ϕ_k is exponentially stable. Moreover, it holds that

$$E\|\phi_k\|^2 \le \frac{\nu}{\mu} E\|\phi_0\|^2 (1-\psi)^k + \frac{\lambda}{\mu\psi} \qquad (21)$$

Proof: This result is a combination of *Lemma 3* and *[19, Th. 2]*

Lemma 2: If $V(\phi_k)$ satisfies

$$E[V(\phi_{k+1}) \mid \phi_k, \phi_{k-1}, \ldots \phi_0] - \frac{V(\phi_k)}{\theta} - L < 0 \qquad (22)$$

for some $\theta > 0$, *L>0*, then $V(\phi_k)$ is bounded with probability 1 and, moreover, $EV(\phi_k)$ remains bounded for all *k* with

$$E[V(\phi_k)] < \frac{V(\phi_0)}{\theta^k} + L\frac{\theta}{\theta-1}\left\{1-\frac{1}{\theta^{k+1}}\right\} \quad (23)$$

Proof : See[20].

Theorem 2 (Exponential Stability) : Given scalars $\varepsilon_1 > 0,\ \varepsilon_2 > 0,$ and $0<\alpha<1$, let $\{A_f, B_f, P_1, P_2, R_1, R_2, R_3, \sigma_1, \sigma_2\}$ be a solution to (7) and (11)-(13) with $\Gamma > I$. Then the resulting process $\{\phi_k, \eta_k\}$ are exponentially stable in the presence of measurement and process noises and for fixed $(A, A_d \text{ and } C)$.

Moreover, the variance of ϕ_k is bounded as follows:

$$E\|\phi_k\|^2 < \frac{1}{\lambda_{\min}(\Gamma)}\left\{\frac{V(\phi_0)}{\theta^k} + L\frac{\theta}{\theta-1}(1-\frac{1}{\theta^{k+1}})\right\} \quad (24)$$

where

$$L \overset{\Delta}{=} \rho_u Tr(B^T(P_1+P_2)B) + \rho_\upsilon Tr(B_f^T P_2 B_f). \quad (25)$$

Proof: The result follows from (17), (18), and *Lemmas 1* and *2*.

Remark: Apart from the above result, we can also show almost sure exponential stability of *(5)* in norm in the absence of noises. Using Chebychev's, in the absence of noises, we have:

$$P\left\{\|\phi_k\| > \frac{1}{\lambda_{\min}(\Gamma)\theta^{\frac{k}{4}}}\right\} \le (\lambda_{\min}(\Gamma)\theta^{\frac{k}{4}})^2 E(\|\phi_k\|^2). \quad (26)$$

Summing over *k*, we get:

$$\sum_{k=0}^{\infty} P\left\{\|\phi_k\| > \frac{1}{\lambda_{\max}(\Gamma)\theta^{\frac{k}{4}}}\right\} \le (\lambda_{\min}(\Gamma))^2 \sum_{k=0}^{\infty}\theta^{\frac{k}{2}} E(\|\phi_k\|^2)$$
$$\le (\lambda_{\min}(\Gamma))^2 \sum_{k=0}^{\infty}\frac{V(\phi_0)}{\theta^{\frac{k}{2}}} = \frac{(\lambda_{\min}(\Gamma))^2 V(\phi_0)}{1-\frac{1}{\theta}^{\frac{1}{2}}} < \infty. \quad (27)$$

Now, from the Borel Cantelli [21], we conclude that the event $\|\phi_k\| \ge (1/\lambda_{\min}(\Gamma)\theta^{k/4})$ cannot occur infinitely often, i.e,

$$P\left\{\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\left\{\|\phi_k\| > \frac{1}{\lambda_{\min}(\Gamma)\theta^{\frac{k}{4}}}\right\}\right\} = 0. \quad (28)$$

Then, it holds that $\|\phi_k\| \ge (1/\lambda_{\min}(\Gamma)\theta^{k/4})$ as desired.

4. POLITOPIC UNCERTAINTIES

We can now incorporate (A, A_d, C) are not fixed but lie within the polytopic set *K* defined by *(3)*.

Theorem 3 (Exponentially-Stable Filter) : Given scalars $\varepsilon_1 > 0,\ \varepsilon_2 > 0,\ \varepsilon_3 > 0$ and $0<\alpha<1$, any filter defined by the matrices

$$A_f = (Q_1 P_2^{-1})^T \text{ and } B_f = (Q_2 P_2^{-1})^T \quad (29)$$

where $Q_1, Q_2, P_2,$ and L_f are obtained from a feasible solution of the matrix inequalities (7) and (11)-(13) for all A taking values in $[A_1......A_p]$, A_d taking values in $[A_1......A_{dp}]$, and C taking values in $[C_1......C_p]$, ensures the following :

i) $E\|\phi_k\|^2$ is bounded as stated in *Theorem 2;*

ii) exponential and asymptotic stability of *(5)* for all admisible parameters (A, A_d, C).

Proof: These properties follow from the definition Q_1 and Q_2, and from the fact that the inequalities (7), (11), and (12) are linear in (A, A_d, C).

The result (24) from Thm. 2 further suggests that we can minimize an upper bound on the error variance, $E\|\tilde{x}_k\|^2$, by seeking filter coeffficients $\{A_f, B_f\}$ that minimize the following function over the variables

$$\{A_f, B_f, P_1, P_2, R, \sigma_1, \sigma_2, \beta\}:$$
$$\rho_u Tr(B^T(P_1+P_2)B) + \rho_v Tr(B_f^T P_2 B_f)$$

subject to conditions (7) and (11)-(13) and $\mathrm{I} < \Gamma$. The last term in the previous cost function is nonlinear in (B_f, P_2). We can instead solve the following convex optimization problem over the variables $\{A_f, B_f, P_1, P_2, R, \Lambda, \sigma_1, \sigma_2, \beta\}$:

$$\min Tr\{\rho_u B^T(P_1+P_2)B + \rho_v \Lambda\} \quad (30)$$

subject to condition (7) and (11)-(13) and

$$\begin{pmatrix} \Lambda & Q_2 \\ Q_2^T & P_2 \end{pmatrix} > 0, \text{ with } \Gamma > I.$$

This last condition enforces a bound $B_f^T P_2 B_f < \Lambda$. Note that since $P > I$, (7): can be enforced by the inequality

$$\begin{pmatrix} \beta I & 0 & A^T P_1 & \hat{J} \\ 0 & \beta I & 0 & Q_1 \\ P_1 A & 0 & P_1 & 0 \\ \hat{J}^T & Q_1^T & 0 & P_2 \end{pmatrix} > 0. \quad (31)$$

5. ROBUST PERFORMANCE

In this section, we shall further assume that

$$E\sum_{k=0}^{\infty} u_k^T u_k < \infty \qquad E\sum_{k=0}^{\infty} v_k^T v_k < \infty. \quad (32)$$

Definition 4 [Robust Performance]: The error system *(5)* will be said to have a robust performance of level $\gamma > 0$ if for all nonzero u_k, v_k, as in (32*)*, it holds for some $\chi > 0$ that

$$E\left\{\sum_{k=0}^{\infty} \bar{z}_k^T \bar{z}\right\} < \chi\|\eta_0\|^2 + \gamma^2 E\left\{\sum_{k=0}^{\infty} u_k^T u_k + v_k^T v_k\right\}.$$

Observe that contrary to a standard H_∞, we use the expectation operator on both sides of the above inequality in order to account for the presence of stochastic uncertainties. In addition to asymptotic and exponential stability, we can enforce a robust performance level by requiring *(P,R)* to satisfy, along with the feasibility conditions (7) and (11)-(13) with $\Gamma > I$ the following requirement:

$$EV(\phi_{k+1}) - EV(\phi_k) - \gamma^2 E(u_k^T u_k + v_k^T v_k) + E\bar{z}_k^T \bar{z}_k < 0 \quad (33)$$

for some given $\gamma > 0$ .Indeed, if we sum (33) over *k*, and noting that the error system is exponentially mean-square stable, we get

$$E\left\{\sum_{k=0}^{\infty} \bar{z}_k^T \bar{z}\right\} < EV(\phi_0) + \gamma^2 E\left\{\sum_{k=0}^{\infty} u_k^T u_k + v_k^T v_k\right\} \quad (34)$$

which is consistent with criterion (33). Now, if β also satisfies

$$\begin{pmatrix} \beta I & \bar{A}^T \\ \bar{A} & I \end{pmatrix} > 0 \quad (35)$$

or, in other words, if $\bar{B}^T \bar{B} < \beta I$, then using the same methodology as in Section *2*, we can verify that (33) is satisfied if (36), holds true for any given $\varepsilon_3 > 0$,

$$\left(\begin{array}{l} S - R_3 \;\; 0\;0\;0 \;\; 0 \;\; \hat{A}^T P_1 \hat{J} \;\; L^T \;\; -L_f^T \\ -R_3^T \;\; P_2 - R_2 - \beta\varepsilon_2^{-1} I \;\; 0\;0\;0\;0\;0 \;\; Q_1 \;\; L_f^T \\ 0\;0\; R_1 - \beta\varepsilon_2^{-1} \;\; R_3 \;\; 0\;0 \;\; A_d^T P_1 \;\; A_d^T P_2 \;\; 0 \\ 0\;0\; R_3^T \;\; R_2 - \beta\varepsilon_2^{-1} I \;\; 0\;\;0\;\;0\;\;0\;\;0 \\ 0\;0\;0\;0\; \gamma^2 I - \beta\varepsilon_3^{-1} \;\; 0 \;\; B^T P_1 \;\; B^T P_2 \;\; 0 \\ 0\;0\;\;0\;\;0\;0\; \gamma^2 I - \beta\varepsilon_3^{-1} \;\; 0 \; -Q_2 \;\; 0 \\ \hline P_1 A \;\; 0 \;\; P_1 A_d \;\; 0 \;\; P_1 B \;\; 0 \; P_1 \;\; 0 \;\; 0 \\ \hat{J}^T \;\; Q_1^T \;\; P_2 A_d \;\; 0 \; P_2 B \; -Q_2^T \;\; 0 \; P_2 \;\; 0 \\ L - L_f \;\; L_f \;\; 0\;\;0\;\;0\;\;0\;\;0\;\;0 \;\; I \end{array}\right) > 0 \quad (36)$$

where

$$\hat{S} \stackrel{\Delta}{=} P_1 - \rho_\Delta G^T E^T (P_1 + P_2) EG - R_1 - \beta\varepsilon_1^{-1} I - (\sigma_1 + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\sigma_2) U^T U.$$

Theorem 4 (robust Performance): Given scalars $\varepsilon_1 > 0,\ \varepsilon_2 > 0,\ \varepsilon_3 > 0$, and $0 < \alpha < 1$, let $\{A_f, B_f, P_1, P_2, R_1, R_2, R_3\}$ be a solution to the linear matrix inequalities (7), (11)-(13), (35), and (36), with $\Gamma > I$. Then the error system (5) is exponentially stable and has a robust performance level of γ for fixed (A, A_d, C).

6. DELAYLESS SYSTEMS

If we assume a delayless system, i.e. we set $A_d = 0$ in (1), and if we drop the robustness requirement of Section 5, we can enforce a tighter upper bound on the variance of the error, $E\left\|\bar{x}\right\|^2$. Thus, consider the system

$$x_{k+1} = (A + \Delta A_k) x_k + B u_k + D f(x_k) \quad (37)$$

$$y_k = C x_k + v_k. \quad (38)$$

Then, we have the following result.

Theorem 5 (Exponential Stability): Given scalars $\varepsilon_1 > 0,\ \varepsilon_2 > 0,$ and $0 < \alpha < 1,$ let $\{A_f, B_f, P_1, P_2, \sigma_1, \sigma_2\}$ be a solution to the following inequalities :

$$\begin{pmatrix} \beta I & \bar{A}^T \\ \bar{A} & I \end{pmatrix} > 0, \quad \begin{pmatrix} \sigma_1 I & D^T P_1 & D^T P_2 \\ P_1 D & P_1 & 0 \\ P_2 D & 0 & P_2 \end{pmatrix} > 0 \quad (39)$$

$$\begin{pmatrix} \sigma_2 I & D^T P_1 & D^T P_2 \\ P_1 D & P_1 & 0 \\ P_2 D & 0 & P_2 \end{pmatrix} > 0 \quad (40)$$

$$\left(\begin{array}{l} S' \;\; 0 \mid A^T P_1 \;\; \hat{J} \\ 0 \;\; P_2 - \varepsilon^{-1}\beta I - \alpha I \mid 0 \;\; Q_1 \\ \hline P_1 A \;\; 0 \mid P_1 \;\; 0 \\ J^T \;\; \hat{Q}_1^T \mid 0 \;\; P_2 \end{array}\right) > 0 \quad (41)$$

with *P>I*, where

$$\hat{S} \stackrel{\Delta}{=} P_1 - \alpha I - \rho_\Delta G^T E^T (P_1 + P_2) EG - \beta\varepsilon^{-1} I.$$

Then, the resulting process $\{\eta_k\}$ is exponentially stable in the presence of measurement and process noises. Moreover, its variance is bounded as follows:

$$E\left\|\eta_k\right\|^2 < \frac{1}{\lambda_{\min}(P)}\left\{\frac{V(\eta_0)}{\theta^k} + L\frac{\theta}{\theta - 1}\left(1 - \frac{1}{\theta^{k+1}}\right)\right\} \quad (42)$$

where

$$L = \rho_u Tr(B^T(P_1 + P_2)B) + \rho_v Tr(B_f^T P_2 B_f) \quad (43)$$

and $\theta = (1)/(1 - \psi)$ cu $\psi = (\alpha)/(\lambda_{\max}(P)).$

7. SIMULATIONS

To illustrate the multi-objective filter developed for state-delayed systems, we choose an implementation of order 2 for a nonlinear uncertain stochastic system (1) as follows:

$$A_1 = \begin{pmatrix} 0.62 & 0 \\ 0 & 0.61 \end{pmatrix} \; A_2 = \begin{pmatrix} 0.5 & -1 \\ 0.2 & 0.5 \end{pmatrix} \; A_3 = \begin{pmatrix} 0.54 & 1 \\ 0 & 0.56 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; C_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0 \end{pmatrix} \; B = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix} \; D = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$$

$$A_d = \begin{pmatrix} 0.1 & 21 \\ 0.31 & 0.1 \end{pmatrix} \; G = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix} \; E = \begin{pmatrix} 0.01 & 0.02 \\ 0.01 & 0.02 \end{pmatrix}$$

$$\begin{pmatrix} x_{1,k+1} \\ x_{2,k+1} \end{pmatrix} = (A + \Delta A_k)\begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + A_d \begin{pmatrix} x_{1,k-\tau} \\ x_{2,k-\tau} \end{pmatrix} + B w_k$$

$$+ D\begin{pmatrix} 0.1\sin(x_{1,k}) \\ 0.1\sin(x_{1,k}) \end{pmatrix} \quad y_k = C\begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + v_k.$$

The delay τ in the example is chosen as *4*. The values of ε_1 and ε_2 are chosen as *1.1.* The value of β is 1. The robustness level is $\gamma = 8$. The performance of the filter is illustrated in *Fig. 1(a)*, wich shows its tracking capability.

To illustrate the robust minimum variance filter developed in Section *VI* for delayless systems, we choose the following model:

$$A_1 = \begin{pmatrix} 0.62 & 0 \\ 0 & 0.61 \end{pmatrix} \; A_2 = \begin{pmatrix} 0.5 & -1 \\ 0.2 & 0.5 \end{pmatrix} \; A_3 = \begin{pmatrix} 0.54 & 1 \\ 0 & 0.56 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 100 & 0 \\ 50 & 10 \end{pmatrix} \; C_2 = \begin{pmatrix} 90 & 0 \\ 50 & 10 \end{pmatrix} \; B = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} \; E = \begin{pmatrix} 0.1 & 0.2 \\ 0.1 & 0.2 \end{pmatrix}$$

$$\begin{pmatrix} x_{1,k+1} \\ x_{2,k+1} \end{pmatrix} = (A + \Delta A_k) \begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + A_d \begin{pmatrix} x_{1,k-\tau} \\ x_{2,k-\tau} \end{pmatrix}$$

$$+ Bw_k + D \begin{pmatrix} 0.1\sin(x_{1,k}) \\ 0.1\sin(x_{1,k}) \end{pmatrix}$$

$$y_k = C \begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + v_k .$$

Fig. 1(b) compares the mean-square-error $E\left\|\hat{x}_k\right\|^2$ in ***db*** when the actual state matrix is A_3, for both the Kalman filter operating at the centroid of the polytopic region and the robust filter. The noise variances are equal to 1.

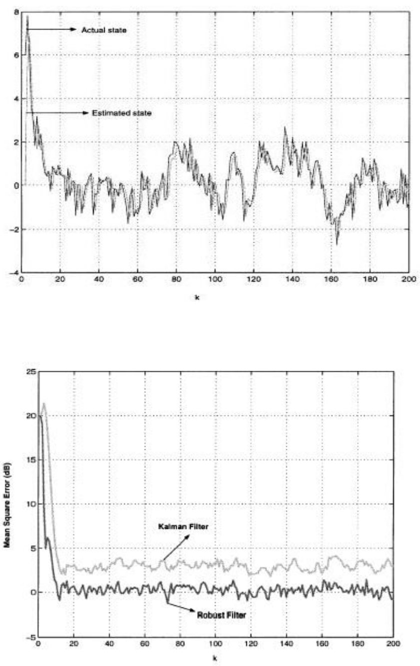

Fig. 1. Performance of the robust filters. a) Tracking performance of the multiobjective robust filter of Theorem 3. b) Mean square error behavior of the Kalman filter and the robust filter for delayless systems of Theorem 5.

8. CONCLUSION

In this note, we developed a multi-objective robust state estimator for uncertain discrete time state-delay systems with mixed deterministic and stochastic uncertainties. The design guarantees almost sure bounded error variance with exponential stability and robust performance.

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