TIME VARIANT NONLINEAR SYSTEM IDENTIFICATION BY DISTRIBUTIONS; APPLICATION FOR WASTEWATER BIODEGRADATION PROCESS

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Abstract: This paper presents a procedure for time variant nonlinear system identification based on distribution theory. Some of the system parameters change in time according to unknown laws. These laws are expressed as finite degree time polynomials whose parameters are included in the set of parameters to be identified. Mainly it is an extension of the procedure developed in (Marin et al., 2005). Considering functionals weighted by polynomials, it is possible to transform a time variant differential system of equations to an algebraic system in unknown parameters. The hierarchical structure identification method for rational expressions, proposed in (Marin et al., 2005), is now extended to time variant systems.
An application for parameter identification of a wastewater biodegradation process, considering time variant yield coefficients is presented.

Keywords**:** Identification; Distribution theory; Functionals, Bioprocesses.

1. INTRODUCTION

As is presented in (Marin et al., 2005), progresses have been made in the area of continuous-time system identification. Many discussions, methods and results on continuous-time identification are presented in (Unbehanen and Rao, 1987), (Li and Billings, 2001), (Landau et al., 2001), (Sinh and Rao, 1991), (Marin, 1992, 1993, 1999); (Bastogne et al., 1997); (Hoverkamp et al., 1996), (Overschee and De Moor, 1996).
A novel approach for continuous-time system identification is that based on distribution theory, using deterministic distributions (Marin, 2002) or random distributions (Ohsumi et al., 2002).
Identification of the non-linear continuous-time systems is far away more complicated. The traditional procedures are based on the Volterra functional series (Schetzen, 1980), expressed in time domain (Boyd and Chua, 1985) or frequency domain (Li and Billings, 2001).
The parameter identification of deterministic nonlinear continuous-time systems (NCTS), modeled by polynomial type differential equation, has been considered by numerous authors, (Pearson and Lee, 1985), (Patra and Unbehauen, 1995).
In (Marin et al., 2005), it is presented a method for identification of nonlinear continuous time systems (NCTS) considering that the unknown parameters can appear in rational relations with measured variables. Using techniques utilized in distribution approach (Marin 1992, 1993, 1999), the measurable functions and their derivatives are represented by functionals on a fundamental space of testing functions. Such systems are common in biotechnology (Bastin and Dochain, 1990), (Petre, 2002), (Selişteanu et al., 2004).
The main idea from (Marin et al., 2005) is to use a hierarchical structure of identification. First, some state equations are utilized to obtain a set of linear equations in some parameters. The results of this first stage of identification are utilized for expressing other parameters by linear equations. This process is repeated until all parameters are identified.
In the present paper, the above idea of hierarchical identification has been extended to time variant systems.

Variable parameters are modeled by finite degree time polynomials whose unknown coefficients are included in the set of parameters to be identified.
To transform a differential time variant system of equations to an algebraic system of functionals, the so-called weighted distributions are considered. Weighted distributions are nothing else rather the product between time functions and distributions.
The paper is organized as follows: The structure of a NCTS describing a wastewater biodegradation process is given in Section 2. Section 3, presents some aspects regarding weighted distributions and their derivatives. The problem statement of continuous time variant system identification is analyzed in Section 4, followed, in Section 5, by presenting the distribution approach of identification. The hierarchical structure of identification and estimation equations takes the space of Section 6. Some experimental results are presented in Section 7, and conclusions in Section 8.

2. MATHEMATICAL MODEL OF WASTE WATER BIODEGRADATION PROCESS

Shall we consider, as in (Marin et al., 2005), the same system represented by the model of a wastewater treatment plant, but, assuming now that the values of some unknown parameters can modify in time. This is a biomethanation process of wastewater biodegradation, producing methane gas. Such reactions take place inside a continuous stirred tank bioreactors whose reduced mathematical models express two phases processes as presented in (Selişteanu et al., 2004), (Bastin and Dochain, 1990), (Petre, 2002). The following simplified reaction scheme is considered,

$$S_1 \xrightarrow{\phi_1} X_1 + S_2 \; ; \; S_2 \xrightarrow{\phi_2} X_2 + P_1 \; ; \quad (1)$$

where: S_1 represents the glucose substrate, S_2 the acetate substrate, X_1 is the acidogenic bacteria, X_2 the acetoclastic methanogenic bacteria and P_1 represents the product, i.e. the methane gas. The reaction rates are denoted by ϕ_1, ϕ_2.
The corresponding dynamical model,(Petre, 2002), is

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ S_1 \\ X_2 \\ S_2 \\ P_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k_1 & 0 \\ 0 & 1 \\ k_2 & -k_3 \\ 0 & k_4 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} - D\begin{bmatrix} X_1 \\ S_1 \\ X_2 \\ S_2 \\ P_1 \end{bmatrix} + \begin{bmatrix} 0 \\ DS_{in} \\ 0 \\ 0 \\ -Q_1 \end{bmatrix} \quad (2)$$

where the state vector of the model is

$$\xi = [X_1 \; S_1 \; X_2 \; S_1 \; P_1]^T = [\xi_1 \; \xi_2 \; \xi_3 \; \xi_4 \; \xi_5]^T \quad (3)$$

whose components are concentrations in (g/l).
The parameters k_1, k_2, k_3, k_4 are the so-called yield coefficients.
In this paper, we consider that the first two yield coefficients change in time, that means,

$$k_1 = k_1(t), \; k_2 = k_2(t)$$

The reaction rates are nonlinear functions of the state components, expressed as

$$\phi = \phi(\xi) = [\phi_1(\xi) \; \phi_2(\xi)]^T . \quad (4)$$

The vector of feed rates and of rates of removal of components is denoted

$$F = [0 \;\; D \cdot S_{in} \;\; 0 \;\; 0 \;\; -Q_1]^T \quad (5)$$

where, D is the dilution rate, a scalar in this particular case, S_{in} represents the concentration of the externally influent substrate–glucose, Q_1 is the methane gas outflow rate.
The dynamical model (2) can be compactly written

$$\frac{d\xi}{dt} = K(t) \cdot \phi(\xi) - D \cdot \xi + F . \quad (6)$$

In fact, this model describes the behavior of an entire class of biotechnological processes. It referees as the general dynamical state-space model of this class of bioprocesses (Bastin and Dochain, 1990). In (6), $K(t)$ is the matrix of the yield coefficients k_{ij}

$$K(t) = \begin{bmatrix} 1 & -k_1(t) & 0 & k_2(t) & 0 \\ 0 & 0 & 1 & -k_3 & k_4 \end{bmatrix}^T \quad (7)$$

The reaction rates for this process are given by the Monod law

$$\phi_1(\xi) = \mu_1 \cdot \frac{S_1 \cdot X_1}{K_{M_1} + S_1} , \quad (8)$$

and the Haldane kinetic model

$$\phi_2(\xi) = \mu_2 \cdot \frac{S_2 \cdot X_2}{K_{M_2} + S_2 + S_2^2 / K_i} , \quad (9)$$

where K_{M_1}, K_{M_2} are Michaelis-Menten constants; μ_1, μ_2 represent specific growth rates coefficients and K_i is the inhibition constant.
Because variation speed of $k_1(t)$, $k_2(t)$ is not so high, their time law variation is locally approximated by straight lines

$$k_1(t) = \theta_{11} \cdot t + \theta_{10} \; , \; \theta_1 = [\theta_{11}, \theta_{10}] \quad (10)$$

$$k_2(t) = \theta_{21} \cdot t + \theta_{20} \; , \; \theta_2 = [\theta_{21}, \theta_{20}] \quad (11)$$

It is considered that all parameters $\theta_{11}, \theta_{10}, \theta_{21}, \theta_{20}$ are constant during the time intervals when identification is performed. During these time intervals, different functionals representing distributions are evaluated to obtain a final result.
In such way, the time curves (10), (11) are piece wise approximated by straight lines.
For simplicity, all parameters characterizing the plant are gathered in a single vector

$$\theta = [\theta_{11} \; \theta_{10} \; \theta_{21} \; \theta_{20} \; \theta_3 \; \theta_4 \; \theta_5 \; \theta_6 \; \theta_7 \; \theta_8 \; \theta_9]^T , \quad (12)$$

where

$$\theta_3 = k_3 ; \theta_4 = k_4 ; \theta_5 = \mu_1 ; \theta_6 = \mu_2 ; \quad (13)$$

$$\theta_7 = K_{M_1} ; \; \theta_8 = K_{M_2} ; \; \theta_9 = K_i . \quad (14)$$

Because the dilution rate D can be externally modified, it will be considered as the third component of the input vector, $u = [u_1 \; u_2 \; u_3]^T$.
The other two components of u are the concentration S_{in} and the methane gas outflow Q_1

$$u_1 = S_{in};\ u_2 = Q_1;\ u_3 = D\,. \qquad (15)$$

Written explicitly by components, the state equations (2) or (6), within the above notations, takes the form,

$$\dot{\xi}_1 = \phi_1 - u_3 \cdot \xi_1 \qquad (16)$$

$$\phi_1 = \theta_5 \cdot \frac{\xi_1 \cdot \xi_2}{\theta_7 + \xi_2} \qquad (17)$$

$$\dot{\xi}_2 = -(\theta_{11} \cdot t + \theta_{10}) \cdot \phi_1 - u_3 \cdot \xi_2 + u_1 \cdot u_3 \qquad (18)$$

$$\dot{\xi}_3 = \phi_2 - u_3 \cdot \xi_3 \qquad (19)$$

$$\phi_2 = \theta_6 \cdot \frac{\xi_3 \cdot \xi_4}{\theta_8 + \xi_4 + \theta_9^{'} \cdot \xi_4^2}\ ,\ \theta_9^{'} = \frac{1}{\theta_9} \qquad (20)$$

$$\dot{\xi}_4 = (\theta_{21} \cdot t + \theta_{20}) \cdot \phi_1 - \theta_3 \cdot \phi_2 - u_3 \cdot \xi_4 \qquad (21)$$

$$\dot{\xi}_5 = -u_3 \cdot \xi_5 + \theta_4 \cdot \phi_2 - u_2 \qquad (22)$$

3. WEIGHTED DISTRIBUTIONS AND THEIR DERIVATIVES

Let us denote by Φ_n the fundamental space from distribution theory (Kecs et al. 1975), of the real fundamental functions,

$$\varphi : \mathbb{R} \to \mathbb{R}, t \to \varphi(t)\,, \qquad (23)$$

with compact support T, having continuous derivatives at least up to the order n. The linear space Φ_n is organized as a topological space considering the norm,

$$\|\varphi\|_n = \sup_{0\le k\le n,\, t\in T} |\varphi^{(k)}(t)| = \max_{0\le k\le n}\{ \sup_{t\in T} |\varphi^{(k)}(t)|\}\,. \qquad (24)$$

A distribution is a linear, continuous (in the above topology) real functional on Φ_n,

$$F : \Phi_n \to \mathbb{R}, \varphi \to F(\varphi) \in \mathbb{R} \qquad (25)$$

Let $\quad q : \mathbb{R} \to \mathbb{R}, t \to q(t) \qquad (26)$

be a function that admits a Riemann integral on any compact interval T from $\mathbb{R}$. Using this function, a unique distribution

$$F_q : \Phi_n \to \mathbb{R},\ \varphi \to F_q(\varphi) \in \mathbb{R} \qquad (27)$$

can be built by the relation

$$F_q(\varphi) = \int_{\mathbb{R}} q(t) \cdot \varphi(t) \cdot dt\,,\ \forall \varphi \in \Phi_n\,. \qquad (28)$$

Considering, at least, $q \in \mathrm{C}^0(\mathbb{R})$, the following important equivalence take place (Barbu, 1985),

$$F_q(\varphi) = 0, \forall \varphi \in \Phi_n \Leftrightarrow q(t) = 0, \forall t \in \mathbb{R}\,. \qquad (29)$$

Let

$$\rho_j : \mathbb{R} \to \mathbb{R}, t \to \rho_j(t),\ \rho_j \in \mathrm{C}^n(\mathbb{R}),\ j \in J \qquad (30)$$

be a set of continuous time functions.

The product between a function (30), ρ_j, and a distribution (27), F_q, is a new distribution (Kecs et al. 1975),

$$F_q^{\rho_j} = \rho_j \cdot F_q : \Phi_n \to \mathbb{R},\ \varphi \to F_q^{\rho_j}(\varphi) \in \mathbb{R},\ j \in J \qquad (31)$$

built through the relation

$$F_q^{\rho_j}(\varphi) = \int_{\mathbb{R}} \rho_j(t) \cdot q(t) \cdot \varphi(t) \cdot dt\,,\ \forall \varphi \in \Phi_n \qquad (32)$$

Under this interpretation, a single function q generates a family of distributions $F_q^{\rho_j}$, $j \in J$ on Φ_n.

Let us denote by $\Phi_n^{/}$ the dual space of Φ_n, which means the set of all distributions defined on Φ_n.

Because φ from (23), (24) has a finite compact support and all functions $\rho_j, \forall j \in J$ of (30) are continuous with derivatives at least up to the order n then the functions

$$\varphi_{\rho_j} = \rho_j \cdot \varphi \in \Phi_n, \forall j \in J\,, \qquad (33)$$

are testing functions on Φ_n, where

$$\varphi_{\rho_j}(t) = \rho_j(t) \cdot \varphi(t), \forall t \in \mathbb{R}, \forall j \in J\,. \qquad (34)$$

If q is a linear combination of continuous time weighted functions $\rho_j(t) \cdot q_i(t)$, $i = 1 : p, j \in J$ then the distribution F_q on Φ_n is a linear combination, on the same space $\Phi_n^{/}$, of the weighted distributions $F_{q_i}^{\rho_j}$, $j \in J$, generated by the components q_i, and the family of weighting functions $\rho_j, j \in J$

$$q = \sum_{i=1}^{p} \alpha_i \cdot \rho_j(t) \cdot q_i(t)_i \Rightarrow \qquad (35)$$

$$F_q(\varphi) = \sum_{i=1}^{p} \alpha_i \cdot F_{q_i}^{\rho_j}(\varphi) \qquad (36)$$

$$F_q^{\rho_j}(\varphi) = \int_{\mathbb{R}} \rho_j(t) \cdot q(t) \cdot \varphi(t) \cdot dt = F_{q_i}(\rho_j \cdot \varphi),\ \forall \varphi \in \Phi_n \qquad (37)$$

where,

$$F_{q_i}(\varphi) = \int_{\mathbb{R}} q_i(t) \cdot \varphi(t) \cdot dt,\ \forall \varphi \in \Phi_n\,. \qquad (38)$$

The notion of distribution k-order derivative, $k = 0 : n$, (Kecs et al. 1975), is now directly extended to weighted distributions.

If $F_q^{\rho_j} \in \Phi_n^{/}$, ρ^j of (30), then its k-order derivative, $k \le n$, is a new distribution, $F_q^{\rho_j(k)} \in \Phi_n^{/}$ uniquely defined by the relations,

$$F_q^{\rho_j(k)}(\varphi) = (-1)^k \cdot F_q([\rho_j \cdot \varphi]^{(k)}), \forall \varphi \in \Phi_n \qquad (39)$$

where

$$[\rho_j \cdot \varphi]^{(k)} = \frac{d^k[\rho_j(t) \cdot \varphi(t)]}{dt^k} = \sum_{i=1}^{k} \mathrm{C}_{\mathrm{k}}^{\mathrm{i}} \cdot \rho_j^{(i)} \cdot \varphi^{(k-i)} \qquad (40)$$

is the k-order ($k = 0 : n$) time derivative of the testing function $\varphi_{\rho_j} = \rho_j \cdot \varphi \in \Phi_n$, and

$$\varphi^{(k)} : \mathbb{R} \to \mathbb{R},\ t \to \varphi^{(k)}(t) = \frac{d^k \varphi(t)}{dt^k} \qquad (41)$$

$$\rho_j^{(i)} : \mathbb{R} \to \mathbb{R},\ t \to \rho_j^{(i)}(t) = \frac{d^i \rho_j(t)}{dt^i}\,. \qquad (42)$$

Taking into account (40), (39) can be expressed as

$$F_q^{\rho_j(k)}(\varphi) = (-1)^k \cdot \sum_{i=1}^{k} \mathrm{C}_{\mathrm{k}}^{\mathrm{i}} \cdot F_q^{\rho_j^{(i)}}(\varphi^{(k-i)}), \forall \varphi \in \Phi_n \qquad (43)$$

where,

$$\varphi \to F_q^{\rho_j^{(i)}}(\varphi) = \int_{\mathbb{R}} q(t) \cdot [\rho_j^{(i)}(t) \cdot \varphi(t)] \cdot dt \in \mathbb{R} \qquad (44)$$

is a weighted distribution generated by function q. When $q \in C^k(\mathbb{R})$,then $F_q^{\rho_j(k)} = F_{q^{(k)}}^{\rho_j}$

$$F_q^{\rho_j(k)}(\varphi) = F_{q^{(k)}}^{\rho_j}(\varphi) = \int_{\mathbb{R}} q^{(k)}(t) \cdot \rho_j(t) \cdot \varphi(t) dt \qquad (45)$$

that means the k-order derivative of a weighted distribution generated by a function $q \in \mathrm{C}^k(\mathbb{R})$ equals to the weighted distribution generated by $q^{(k)}$, the k-order time derivative of the function q ,

$$q^{(k)} : \mathbb{R} \to \mathbb{R}, t \to q^{(k)}(t) = \frac{d^k q(t)}{dt^k}. \qquad (46)$$

If $q \in \mathrm{C}^k(\mathbb{R})$, from (39), (45) one can write, $\forall \varphi \in \Phi_n$

$$F_q^{\rho_j(k)}(\varphi) = \int_{\mathbb{R}} q^{(k)}(t) \cdot \rho_j(t) \cdot \varphi(t) \cdot dt = \qquad (47)$$

$$= (-1)^k \cdot \sum_{i=1}^{k} \mathrm{C}_\mathrm{k}^\mathrm{i} \cdot \int_{\mathbb{R}} q(t) \cdot [\rho_j^{(i)}(t) \cdot \varphi^{(k-i)}(t)] \cdot dt, \forall \varphi \in \Phi_n$$

For example, $\rho_j(t) = t \;\Rightarrow$ (48)

$$F_q^{t\,(k)}(\varphi) = (-1)^k \cdot [F_q^t(\varphi^{(k)}) + k \cdot F_q^1(\varphi^{(k-1)})], \forall \varphi \in \Phi_n$$

4. PROBLEM STATEMENT OF CONTINUOUS TIME VARIANT SYSTEM IDENTIFICATION

Let us consider a dynamical continuous time variant system with n_u inputs,

$$u : \mathbb{R} \to \mathbb{R}^{n_u}, t \to u(t)\ ,\ u \in \Omega \qquad (49)$$

and n_y outputs,

$$y : \mathbb{R} \to \mathbb{R}^{n_y}, t \to y(t)\ ,\ y \in \Gamma \qquad (50)$$

where Ω represents the set of admissible inputs and Γ is the set of possible outputs.
It can be expressed by a differential operator,

$$q_{\theta/(u,y)} = Q(u, y, t, \theta) \qquad (51)$$

whose expression depends on a vector of parameters

$$\theta = [\theta_1 \ldots \theta_i \ldots \theta_p]^T. \qquad (52)$$

A triple (u^*, y^*, θ^*) is a realization of the model if the function

$$q_{\theta^*/(u^*,y^*)} = Q(u^*, y^*, t, \theta^*) \qquad (53)$$

is the zero function, $q_{\theta^*/(u^*,y^*)} = 0$ that means,

$$q_{\theta^*/(u^*,y^*)}(t) = Q(u^*(t), y^*(t), t, \theta^*) = 0,\ \forall t \in \mathbb{R}. \qquad (54)$$

The value $\theta = \theta^*$ is consistent with the model (53) if and only if the two following conditions are accomplished:

1. Covering condition

$$q_{\theta^*/(u,y)}(t) = Q(u(t), y(t), \theta^*) = 0,\ \forall t \in \mathbb{R}\ , \forall (u, y) \in \Omega \times \Gamma \qquad (55)$$

2. Uniqueness condition

$$q_{\theta/(u,y)}(t) = 0,\ \forall t \in \mathbb{R}\ , \forall (u, y) \in \Omega \times \Gamma \Rightarrow \theta = \theta^* \qquad (56)$$

A special case is the model (51) expressing a linear relation in the parameters

$$q_{\theta/(u,y)} = Q(u, y, t, \theta) = \sum_{i=1}^{p} w_i \cdot \theta - v = w^T \cdot \theta - v, \qquad (57)$$

where w_i and v represent a sum of the time weighted derivatives of some known, possible nonlinear, functions ψ_i^j, ψ_0^j, with respect to the input and output variables,

$$w_i = \sum_{j=1}^{p_i} \rho_i^j(t) \cdot [\psi_i^j(u, y)]^{(n_i^j)}, i = 1 : p, \qquad (58)$$

$$v = \sum_{j=1}^{p_0} \rho_0^j(t) \cdot [\psi_0^j(u, y)]^{(n_0^j)}. \qquad (59)$$

The weighting functions $\rho_i^j(t),\ i = 0 : p,\ j = 1 : p_i$ are given time functions, depending how the time variation of the coefficients is parameterized.
In this paper, considering parameterization by polynomials, these functions are of the form

$$\rho_i^j(t) = t^{m_i^j},\ i = 0 : p,\ j = 1 : p_i. \qquad (60)$$

Parameters $p_i, n_i^j, p_0, n_0^j, m_i^j$ are given integer numbers. The identification problem, into condition (57), has a unique solution. At any time instant t, for a measured input-output pair of functions (u, y) the value of the function $q_{\theta/(u,y)}(t)$ is a real vector

$$q_{\theta/(u,y)}(t) = \sum_{i=1}^{p} w_i(t) \cdot \theta - v(t) = w^T(t) \cdot \theta - v(t) \qquad (61)$$

where

$$w^T(t) = [w_1(t), \ldots, w_i(t), \ldots, w_p(t)],$$

$$w_i(t) = \sum_{j=1}^{p_i} \rho_i^j(t) \cdot [\psi_i^j(u(t), y(t))]^{(n_i^j)}, i = 1 : p \qquad (62)$$

$$v(t) = \sum_{j=1}^{p_0} \rho_0^j(t) \cdot [\psi_0^j(u(t), y(t))]^{(n_0^j)} \qquad (63)$$

Practically, it is possible to record the functions (u, y) in the time interval $T \subset \mathbb{R}$ only, called observation time interval or just time window. The restriction of the functions (u, y) to the time interval T is denoted by (u_T, y_T) respectively. If no confusion would appear, then we may drop the subscript T.

As in (Marin et al., 2005), an identification problem means to determine the parameter $\theta = \widehat{\theta}$, given the priori information on the model structure Q, (51), and the observed input-output pair (u_T, y_T),

$$\widehat{\theta} = \widehat{\theta}(u_T, y_T, Q) \qquad (64)$$

in a such a way that,

$$q_{\widehat{\theta}/(u_T,y_T)}(t) = 0, \forall t \in \mathbb{R} \qquad (65)$$

This condition involves,

$$q_{\widehat{\theta}/(u,y)}(t) = 0, \forall t \in \mathbb{R},\ \forall (u, y) \in \Omega \times \Gamma \qquad (66)$$

for any input-output pair (u, y) observed to that system. As the unknown parameter θ has a finite number p of components, then it will be enough to choose (utilize) a finite number N of time instants, $t_i, i = 1 : N$ based on which to create an algebraic equation. In the specific case of (48), this is a linear system

$$\mathbf{W} \cdot \theta = \mathbf{v} \qquad (67)$$

where $\mathbf{W}$ is a $N \times p$ matrix of real numbers,

$$\mathbf{W} = [w(t_1)^T;..;w(t_i)^T;..;w(t_N)^T], \quad (68)$$

whose i-th row ($i = 1:N$) is,

$$w(t_i)^T = [w_1(t_i),..,w_k(t_i),..,w_p(t_i)] \quad (69)$$

The symbol $\mathbf{v}$ denotes a N column real vector,

$$\mathbf{v} = [v(t_1),..,v(t_i),..,v(t_N)]^T. \quad (70)$$

Let us denote $r = rank(\mathbf{W})$. If $r = p$, then a unique solution is obtained,

$$\widehat{\theta} = (\mathbf{W}^T \cdot \mathbf{W})^{-1} \cdot \mathbf{W}^T \cdot v = \theta^* \quad (71)$$

The equation (71) is of no practical interest because it is not recommended to measure (or to estimate) the derivatives of signals, mainly when they are noise contamined.

5. DISTRIBUTION APPROACH OF TIME VARIANT SYSTEM IDENTIFICATION

Now let us consider known the set of continuous time scalar functions (62), (63)

$$w_i(t) = \sum_{j=1}^{p_i} \rho_i^j(t) \cdot [\psi_i^j(u(t), y(t))]^{(n_i^j)} =$$

$$= \sum_{j=1}^{p_i} \rho_i^j(t) \cdot [\psi_i^j(t)]^{(n_i^j)}, i = 1:p \quad (72)$$

$$v(t) = \sum_{j=1}^{p_0} \rho_0^j(t)[\psi_0^j(u(t), y(t))]^{(n_0^j)} = \sum_{j=1}^{p_0} \rho_0^j(t)[\psi_0^j(t)]^{(n_0^j)} \quad (73)$$

The function $w_i(t)$ from (72) generates a distribution

$$F_{w_i} : \Phi_n \to \mathbb{R}, \varphi \to F_{w_i}(\varphi) \quad (74)$$

as a sum of weighted distributions,

$$F_{w_i}(\varphi) = \sum_{j=1}^{p_i} F^{\rho_i^j}_{[\psi_i^j(t)]^{(n_i^j)}}(\varphi) \quad (75)$$

where

$$F^{\rho_i^j}_{[\psi_i^j(t)]^{(n_i^j)}}(\varphi) = F^{\rho_i^j \, (n_i^j)}_{[\psi_i^j(t)]}(\varphi) =$$

$$= (-1)^{n_i^j} \cdot \sum_{k=1}^{n_i^j} \mathrm{C}^{\mathrm{k}}_{n_i^j} \cdot F^{[\rho_i^j]^{(k)}}_{[\psi_i^j(t)]}(\varphi^{(n_i^j - k)}), \forall \varphi \in \Phi_n \quad (76)$$

$$F^{[\rho_i^j]^{(k)}}_{[\psi_i^j(t)]}(\varphi^{(n_i^j - k)}) = \int_{\mathbb{R}} [\psi_i^j(t)] \cdot [\rho_i^j]^{(k)} \cdot \varphi^{(n_i^j - k)}(t) \cdot dt \quad (77)$$

They constitute the row vector,

$$F_w^T(\varphi) = [F_{w_1}(\varphi),..., F_{w_i}(\varphi),..., F_{w_p}(\varphi)] \in \mathbb{R}^p. \quad (78)$$

Also, the function $v(t)$ from (73) generates a distribution

$$F_v : \Phi_n \to \mathbb{R}, \varphi \to F_v(\varphi) \quad (79)$$

as a weighted distribution,

$$F_v(\varphi) = F^{\rho_0^j}_{[\psi_0^j(t)]^{(n_0^j)}}(\varphi) \quad (80)$$

where

$$F^{\rho_0^j}_{[\psi_0^j(t)]^{(n_0^j)}}(\varphi) = F^{\rho_0^j \, (n_0^j)}_{[\psi_0^j(t)]}(\varphi) \quad (81)$$

$$= (-1)^{n_0^j} \cdot \sum_{k=1}^{n_0^j} \mathrm{C}^{\mathrm{k}}_{n_0^j} \cdot F^{[\rho_0^j]^{(k)}}_{[\psi_0^j(t)]}(\varphi^{(n_0^j - k)}), \forall \varphi \in \Phi_n \quad (82)$$

$$F^{[\rho_0^j]^{(k)}}_{[\psi_0^j(t)]}(\varphi^{(n_0^j - k)}) = \int_{\mathbb{R}} [\psi_0^j(t)] \cdot [\rho_0^j]^{(k)} \cdot \varphi^{(n_0^j - k)}(t) \cdot dt \quad (83)$$

Into this conditions, any input-output pair (u, y) observed from the system (51) is described by a pair of regular distribution (F_w, F_v) for any $\varphi \in \Phi_n$.

In such a way, the problem of identification regarding the parameters of the real system (51) can be represented by distributions. For example, the regular distribution generated by the continuous function $q_{\theta/(u,y)}$ from (51), into the specific case of (61) is related to the parameter vector θ as

$$F_{q_\theta}(\varphi) = F_{q_\theta/(u,y)}(\varphi) = \sum_{i=1}^{p} F_{w_i}(\varphi) \cdot \theta_i - F_v(\varphi) =$$

$$= F_w^T(\varphi) \cdot \theta - F_y(\varphi), \varphi \in \Phi_n \quad (84)$$

If a triple (u^*, y^*, θ^*) is a realization of the model (51), then the identity (85) takes place,

$$F_{q_{\theta^*}}(\varphi) = F_{q_{\theta^*}/(u^*, y^*)}(\varphi) = 0, \forall \varphi \in \Phi_n \quad (85)$$

and vice versa, if an input-output pair (u^*, y^*) of the family of models (51), with unknown parameter θ, generates a distribution

$$F_{q_\theta}(\varphi) = F_{q_\theta/(u^*, y^*)}(\varphi) = \sum_{i=1}^{p} F_{w_i}(\varphi) \cdot \theta_i - F_v(\varphi) \quad (86)$$

which satisfies

$$F_{q_\theta}(\varphi) = F_{q_\theta/(u^*, y^*)}(\varphi) = 0, \forall \varphi \in \Phi_n, \quad (87)$$

then $\theta = \theta^*$.

As θ has p components it is enough a chose (utilize) a finite number $N \geq p$ of fundamental function $\varphi_i, i = 1:N$ and to build an algebraic equation,

$$\mathbf{F}_w \cdot \theta = \mathbf{F}_v \quad (88)$$

where $\mathbf{F_w}$ is an ($N \times p$) matrix of real numbers

$$\mathbf{F}_w = [F_w^T(\varphi_1);...;F_w^T(\varphi_i);...;F_w^T(\varphi_N)]^T \quad (89)$$

where i-th row $F_w^T(\varphi_i)$ is given by (78).

The symbol $\mathbf{F}_v$ denotes an N-column real vector built from (80)-(83),

$$\mathbf{F}_v = [F_v(\varphi_1),..., F_v(\varphi_i),..., F_v(\varphi_N)]^T. \quad (90)$$

When only the restriction (u_T, y_T) of the pair (u, y) on the time interval T, is available, then one must chose φ_i such that

$$\mathrm{supp}(\varphi_i) \subset T \subset \mathbb{R}, \; i = 1:N \quad (91)$$

If $r = rank(\mathbf{F}_w) = p$, then a unique solution is obtained.

$$\widehat{\theta} = (\mathbf{F}_w^T \cdot \mathbf{F}_w)^{-1} \cdot \mathbf{F}_w^T \cdot \mathbf{F}_v = \theta^* \quad (92)$$

6. THE HIERARCHICAL STRUCTURE OF IDENTIFICATION AND ESTIMATION EQUATIONS

The procedure of hierarchical structure identification developed in (Marin et al., 2005) is retaken here considering time variant coefficients (10) and (11).

To obtain linear equations in unknown parameters, the identification problem is split in several simpler interlinked identification problems called identification layers.

Based on the specific structure of this system, it is possible to group the state equations, in such way to determine five interconnected identification problems of the type (92), labeled Layer_*, *=a, b, c, d, e. They are organized in a hierarchical structure. First, in Layer_a, some state equations are utilized to obtain a set of linear equations in some parameters. The results of this first stage of identification are utilized for expressing other parameters by linear equations in Layer_b. This process is repeated in the other layers until all parameters are identified. For each identification layer, the same type of procedures and numerical algorithms are applied.

Layer_a: Identification of $[\theta_{11},\theta_{10}], \theta_1(t)=\theta_{11}\cdot t+\theta_{10}$

Substituting expression ϕ_1 from (16) into (18) we obtain, the Layer_a model (51)

$$q^a_{\theta/(u,y)}=(\dot{\xi}_1+u_3\cdot\xi_1)[\theta_{11}\cdot t+\theta_{10}]-(-\dot{\xi}_2-u_3\cdot\xi_2+u_1\cdot u_3) \quad (93)$$

characterized by

$$\theta^a=[\theta^a_1\ \ \theta^a_2]=[\theta_{11}\ \ \theta_{10}]\ ,\ p^a=2$$

$$w^a_1(t)=t\cdot[\xi_1]^{(1)}+t\cdot[u_3\cdot\xi_1]^{(0)}=\sum_{j=1}^{2}\rho^j_1(t)\cdot[\psi^j_1(t)]^{(n^j_1)}$$

$$w^a_2(t)=1\cdot[\xi_1]^{(1)}+1\cdot[u_3\cdot\xi_1]^{(0)}=\sum_{j=1}^{2}\rho^j_2(t)\cdot[\psi^j_2(t)]^{(n^j_2)}$$

$$v^a(t)=1\cdot[-\xi_2]^{(1)}+1\cdot[-u_3\cdot\xi_2]^{(0)}+1\cdot[u_1\cdot u_3]^{(0)}$$

$$v^a(t)=\sum_{j=1}^{3}\rho^j_0(t)\cdot[\psi^j_0(t)]^{(n^j_0)}$$

$$F^a_{w_1}(\varphi)=\int_{\mathbb{R}}[-\xi_1(t)]\cdot t\cdot\varphi^{(1)}(t)dt+\int_{\mathbb{R}}[-\xi_1(t)]\cdot 1\cdot\varphi^{(0)}(t)dt+$$
$$+\int_{\mathbb{R}}[u_3(t)\xi_1(t)]\cdot t\cdot\varphi^{(0)}(t)dt$$

$$F^a_{w_2}(\varphi)=\int_{\mathbb{R}}[-\xi_1(t)]\cdot 1\cdot\varphi^{(1)}(t)dt+\int_{\mathbb{R}}[u_3(t)\xi_1(t)]\cdot 1\cdot\varphi^{(0)}(t)dt$$

$$F^{a,T}_w(\varphi)=[F^a_{w_1}(\varphi)\ \ F^a_{w_2}(\varphi)]$$

Also,

$$F^a_v(\varphi)=\int_{\mathbb{R}}[\xi_2(t)]\cdot 1\cdot\varphi^{(1)}(t)dt+$$
$$+\int_{\mathbb{R}}[-u_3(t)\xi_2(t)]\cdot 1\cdot\varphi^{(0)}(t)dt+\int_{\mathbb{R}}[u_1(t)u_3(t)]\cdot 1\cdot\varphi^{(0)}(t)dt$$

$$\mathbf{F}^a_w=[F^{a,T}_w(\varphi_1);...;F^{a,T}_w(\varphi_i);...;F^{a,T}_w(\varphi_{N^a})]^T$$

$$\mathbf{F}^a_v=[F^a_v(\varphi_1),...,F^a_v(\varphi_i),...,F^a_w(\varphi_{N^a})]^T$$

$$\widehat{\theta}^a=(\mathbf{F}^{a,T}_w\cdot\mathbf{F}^a_w)^{-1}\cdot\mathbf{F}^{a,T}_w\cdot\mathbf{F}^a_v \quad (94)$$

$$\widehat{\theta}_{11}=\widehat{\theta}^a_1;\ \widehat{\theta}_{10}=\widehat{\theta}^a_2,\ \widehat{\theta}_1(t)=\widehat{\theta}_{11}\cdot t+\widehat{\theta}_{10} \quad (95)$$

Layer_b: Identification of θ_5,θ_7.

Considering known

$$\theta_1(t)=\widehat{\theta}_1(t)=\widehat{\theta}_{11}\cdot t+\widehat{\theta}_{10} \quad (96)$$

from the Layer_a, and substituting (17), equation (18) becomes,

$$\dot{\xi}_2=-\widehat{\theta}_1(t)\cdot\theta_5\cdot\frac{\xi_1\cdot\xi_2}{\theta_7+\xi_2}-u_3\cdot\xi_2+u_1\cdot u_3$$

The Layer_b model (40) is now,

$$q^b_{\theta/(u,y)}=(\xi_1\cdot\xi_2\cdot\widehat{\theta}_1(t))\cdot\theta_5+(\dot{\xi}_2+u_3\cdot\xi_2-u_1\cdot u_3)\cdot\theta_7-$$
$$-(-\xi_2\cdot\dot{\xi}_2-u_3\cdot\xi_2^2+u_1\cdot u_3\cdot\xi_2) \quad (97)$$

characterized by

$$\theta^b=[\theta^b_1\ \ \theta^b_2]=[\theta_5\ \ \theta_7]\ ,\ p^b=2$$

$$w^b_1(t)=[\xi_1\cdot\xi_2\cdot\widehat{\theta}_1(t)]^{(0)}$$

$$w^b_2(t)=[\xi_2]^{(1)}+[u_3\cdot\xi_2]^{(0)}-[u_1\cdot u_3]^{(0)}$$

$$v^b(t)=[-\frac{1}{2}\xi_2^2]^{(1)}+[-u_3\cdot\xi_2^2]^{(0)}+[u_1\cdot u_3\cdot\xi_2]^{(0)}$$

$$F^b_{w_1}(\varphi)=\int_{\mathbb{R}}[\xi_1(t)\cdot\xi_2(t)\cdot\widehat{\theta}_1(t)]\cdot\varphi^{(0)}(t)\cdot dt$$

$$F^b_{w_2}(\varphi)=\int_{\mathbb{R}}[-\xi_2(t)]\cdot\varphi^{(1)}(t)\cdot dt+\int_{\mathbb{R}}[u_3(t)\cdot\xi_2(t)]\cdot\varphi^{(0)}(t)\cdot dt+$$
$$+\int_{\mathbb{R}}[-u_1(t)\cdot u_3(t)]\cdot\varphi^{(0)}(t)\cdot dt$$

$$F^{b,T}_w(\varphi)=[F^b_{w_1}(\varphi)\ \ F^b_{w_2}(\varphi)]$$

Also,

$$F^b_v(\varphi)=\int_{\mathbb{R}}[\frac{1}{2}\cdot\xi_2^2(t)]\cdot\varphi^{(1)}(t)\cdot dt+$$
$$+\int_{\mathbb{R}}[-u_3(t)\cdot\xi_2^2(t)]\cdot\varphi^{(0)}(t)\cdot dt+$$
$$+\int_{\mathbb{R}}[u_1(t)\cdot u_3(t)\cdot\xi_2(t)]\cdot\varphi^{(0)}(t)\cdot dt$$

$$\mathbf{F}^b_w=[F^{b,T}_w(\varphi_1);...;F^{b,T}_w(\varphi_i);...;F^{b,T}_w(\varphi_{N^b})]^T$$

$$\mathbf{F}^b_v=[F^b_v(\varphi_1),...,F^b_v(\varphi_i),...,F^b_w(\varphi_{N^b})]^T$$

$$\widehat{\theta}^b=(\mathbf{F}^{b,T}_w\cdot\mathbf{F}^b_w)^{-1}\cdot\mathbf{F}^{b,T}_w\cdot\mathbf{F}^b_v \quad (98)$$

$$\widehat{\theta}_5=\widehat{\theta}^b_1;\widehat{\theta}_7=\widehat{\theta}^b_2 \quad (99)$$

Layer_c: Identification of $\theta_{21},\theta_{20},\theta_3$

$$\theta_2(t)=\theta_{21}\cdot t+\theta_{20} \quad (100)$$

Considering known $\theta_5=\widehat{\theta}_5;\theta_7=\widehat{\theta}_7$ from the Layer_b the estimated expression $\widehat{\phi}_1$, of the rational ϕ_1, is

$$\widehat{\phi}_1=\widehat{\theta}_5\cdot\frac{\xi_1\cdot\xi_2}{\widehat{\theta}_7+\xi_2} \quad (101)$$

whose time expression is

$$\widehat{\phi}_1(t)=\widehat{\theta}_5\cdot\frac{\xi_1(t)\cdot\xi_2(t)}{\widehat{\theta}_7+\xi_2(t)}.$$

Substituting expression ϕ_2 from (19) and (101) instead of ϕ_1 into (21) we obtain,

$$\dot{\xi}_4=[\theta_{21}\cdot t+\theta_{20}]\cdot\widehat{\phi}_1-\theta_3\cdot[\dot{\xi}_3+u_3\cdot\xi_3]-u_3\cdot\xi_4$$

which determines the Layer_c model (51)

$$q^c_{\theta/(u,y)}=(t\cdot\widehat{\phi}_1)\cdot\theta_{21}+(\widehat{\phi}_1)\cdot\theta_{20}+(-\dot{\xi}_3-u_3\cdot\xi_3)\cdot\theta_3-$$
$$-(\dot{\xi}_4+u_3\cdot\xi_4) \quad (102)$$

characterized by

$$\theta^c=[\theta^c_1\ \ \theta^c_2\ \ \theta^c_3]=[\theta_{21}\ \ \theta_{20}\ \ \theta_3]\ ,\ p^c=3$$

$$w^c_1(t)=[t\cdot\widehat{\phi}_1]^{(0)}$$

$$w^c_2(t)=[\widehat{\phi}_1]^{(0)}$$

$$w^c_3(t)=[-\xi_3]^{(1)}+[-u_3\cdot\xi_3]^{(0)}$$

$$v^c(t)=[-\xi_4]^{(1)}+[u_3\cdot\xi_4]^{(0)}$$

$$F^c_{w_1}(\varphi)=\int_{\mathbb{R}}[t\cdot\widehat{\phi}_1(t)]\cdot\varphi^{(0)}(t)\cdot dt$$

$$F^c_{w_2}(\varphi)=\int_{\mathbb{R}}[\widehat{\phi}_1(t)]\cdot\varphi^{(0)}(t)\cdot dt$$

$$F^c_{w_3}(\varphi)=\int_{\mathbb{R}}[\xi_3(t)]\cdot\varphi^{(1)}(t)\cdot dt+\int_{\mathbb{R}}[-u_3(t)\cdot\xi_3(t)]\cdot\varphi^{(0)}(t)\cdot dt$$

$$F^{c,T}_w(\varphi)=[F^c_{w_1}(\varphi)\ \ F^c_{w_2}(\varphi)\ \ F^c_{w_3}(\varphi)]$$

Also,

$$F^c_v(\varphi)=\int_{\mathbb{R}}[-\xi_4(t)]\cdot\varphi^{(1)}(t)\cdot dt+$$

$$+\int_{\mathbb{R}}[u_3(t)\cdot\xi_4(t)]\cdot\varphi^{(0)}(t)\cdot dt$$

$$\mathbf{F}^c_w=[F^{c,T}_w(\varphi_1);...;F^{c,T}_w(\varphi_i);...;F^{c,T}_w(\varphi_{N^c})]^T$$

$$\mathbf{F}^c_v=[F^c_v(\varphi_1),...,F^c_v(\varphi_i),...,F^c_w(\varphi_{N^b})]^T$$

$$\widehat{\theta}^c=(\mathbf{F}^{c,T}_w\cdot\mathbf{F}^c_w)^{-1}\cdot\mathbf{F}^{c,T}_w\cdot\mathbf{F}^c_v \tag{103}$$

$$\widehat{\theta}_{21}=\widehat{\theta}^c_1;\widehat{\theta}_{20}=\widehat{\theta}^c_2;\widehat{\theta}_3=\widehat{\theta}^c_3 \tag{104}$$

$$\widehat{\theta}_2(t)=\widehat{\theta}_{21}\cdot t+\widehat{\theta}_{20} \tag{105}$$

Layer_d: Identification of $\theta_6,\theta_8,\theta'_9$.

Considering known $\theta_2=\widehat{\theta}_2(t)$, $\theta_3=\widehat{\theta}_3$ from the Layer_c, and substituting (20) in equation (21) where ϕ_1 is replaced by $\widehat{\phi}_1$ we obtain,

$$\dot{\xi}_4=\widehat{\theta}_2(t)\cdot\widehat{\phi}_1-\widehat{\theta}_3\cdot\frac{\xi_3\cdot\xi_4}{\theta_8+\xi_4+\theta'_9\cdot\xi_4^2}-u_3\cdot\xi_4$$

The Layer_d model (51) is now,

$$q^d_{\theta/(u,y)}=(\xi_3\cdot\xi_4\cdot\widehat{\theta}_3)\cdot\theta_6+(\dot{\xi}_4+u_3\cdot\xi_4-\widehat{\theta}_2(t)\cdot\widehat{\phi}_1)\cdot\theta_8+$$
$$+(\xi_4^2\cdot\dot{\xi}_4+u_3\cdot\xi_4^3-\widehat{\theta}_2(t)\cdot\widehat{\phi}_1\cdot\xi_4^2)\cdot\theta'_9-$$
$$-(-\xi_4\cdot\dot{\xi}_4-u_3\cdot\xi_4^2+\widehat{\theta}_2(t)\cdot\widehat{\phi}_1\cdot\xi_4) \tag{106}$$

characterized by

$$\theta^d=[\theta^d_1\ \ \theta^d_2\ \ \theta^d_3]=[\theta_6\ \ \theta_8\ \ \theta'_9],\ \ p^d=3$$

$$w^d_1(t)=[\xi_3\cdot\xi_4\cdot\widehat{\theta}_3]^{(0)}$$

$$w^d_2(t)=[\xi_4]^{(1)}+[u_3\cdot\xi_4]^{(0)}+[-\widehat{\theta}_2(t)\cdot\widehat{\phi}_1]^{(0)}$$

$$w^d_3(t)=[\frac{1}{3}\xi_4^3]^{(1)}+[u_3\cdot\xi_4^3]^{(0)}+[-\widehat{\theta}_2(t)\cdot\widehat{\phi}_1\cdot\xi_4^2]^{(0)}$$

$$v^d(t)=[-\frac{1}{2}\xi_4^2]^{(1)}+[-u_3\cdot\xi_4^2]^{(0)}+[\widehat{\theta}_2(t)\cdot\widehat{\phi}_1\cdot\xi_4]^{(0)}$$

$$F^d_{w_1}(\varphi)=\int_{\mathbb{R}}[\xi_3(t)\cdot\xi_4(t)\cdot\widehat{\theta}_3]\cdot\varphi^{(0)}(t)\cdot dt$$

$$F^d_{w_2}(\varphi)=\int_{\mathbb{R}}[-\xi_4(t)]\cdot\varphi^{(1)}(t)\cdot dt+\int_{\mathbb{R}}[u_3(t)\cdot\xi_4(t)]\cdot\varphi^{(0)}(t)\cdot dt+$$

$$+\int_{\mathbb{R}}[-\widehat{\theta}_2(t)\cdot\widehat{\phi}_1(t)]\cdot\varphi^{(0)}(t)\cdot dt$$

$$F^{d,T}_w(\varphi)=[F^d_{w_1}(\varphi)\ \ F^d_{w_2}(\varphi)\ \ F^d_{w_3}(\varphi)]$$

Also,

$$F^d_v(\varphi)=\int_{\mathbb{R}}[\frac{1}{2}\cdot\xi_4^2(t)]\cdot\varphi^{(1)}(t)\cdot dt+$$

$$+\int_{\mathbb{R}}[-u_3(t)\cdot\xi_4^2(t)]\cdot\varphi^{(0)}(t)\cdot dt+$$

$$+\int_{\mathbb{R}}[\widehat{\theta}_2(t)\cdot\widehat{\phi}_1(t)\cdot\xi_4(t)]\cdot\varphi^{(0)}(t)\cdot dt$$

$$\mathbf{F}^d_w=[F^{d,T}_w(\varphi_1);...;F^{d,T}_w(\varphi_i);...;F^{d,T}_w(\varphi_{N^d})]^T$$

$$\mathbf{F}^d_v=[F^d_v(\varphi_1),...,F^d_v(\varphi_i),...,F^d_w(\varphi_{N^d})]^T$$

$$\widehat{\theta}^d=(\mathbf{F}^{d,T}_w\cdot\mathbf{F}^d_w)^{-1}\cdot\mathbf{F}^{d,T}_w\cdot\mathbf{F}^d_v \tag{107}$$

$$\widehat{\theta}_6=\widehat{\theta}^d_1;\widehat{\theta}_8=\widehat{\theta}^d_2;\widehat{\theta}'_9=\widehat{\theta}^d_3\Rightarrow\widehat{\theta}_9=1/\widehat{\theta}'_9 \tag{108}$$

Layer_e: Identification of θ_4

Considering known $\theta_6=\widehat{\theta}_6;\theta_8=\widehat{\theta}_8;\theta'_9=\widehat{\theta}'_9$, from the Layer d identification, the estimated expression $\widehat{\phi}_2$, of the nonlinear function ϕ_2, is

$$\widehat{\phi}_2=\widehat{\theta}_6\cdot\frac{\xi_3\cdot\xi_4}{\theta_8+\xi_4+\widehat{\theta}'_9\cdot\xi_4^2} \tag{109}$$

whose time expression is

$$\widehat{\phi}_2=\widehat{\theta}_6\cdot\frac{\xi_3(t)\cdot\xi_4(t)}{\theta_8+\xi_4(t)+\widehat{\theta}'_9\cdot\xi_4^2(t)}.$$

Substituting expression (109) instead of ϕ_2 into (22),

$$\dot{\xi}_5=-u_3\cdot\xi_5+\theta_4\cdot\widehat{\phi}_2-u_2$$

which determines the Layer_e model (40)

$$q^e_{\theta/(u,y)}=(\widehat{\phi}_2)\cdot\theta_4-(\dot{\xi}_5+u_3\cdot\xi_5+u_2) \tag{110}$$

characterized by

$$\theta^e=[\theta^e_1]=[\theta_4],\ \ p^e=1$$

$$w^e_1(t)=[\widehat{\phi}_2]^{(0)};\ \ v^e(t)=[\xi_5]^{(1)}+[u_3\cdot\xi_4]^{(0)}+[u_2]^{(0)}$$

$$F^e_{w_1}(\varphi)=\int_{\mathbb{R}}[\widehat{\phi}_2(t)]\cdot\varphi^{(0)}(t)\cdot dt;\ \ F^{e,T}_w(\varphi)=[F^e_{w_1}(\varphi)]$$

Also,

$$F^e_v(\varphi)=\int_{\mathbb{R}}[-\xi_5(t)]\cdot\varphi^{(1)}(t)\cdot dt+$$

$$+\int_{\mathbb{R}}[u_3(t)\cdot\xi_5(t)]\cdot\varphi^{(0)}(t)\cdot dt+\int_{\mathbb{R}}[u_2(t)]\cdot\varphi^{(0)}(t)\cdot dt$$

$$\mathbf{F}^e_w=[F^{e,T}_w(\varphi_1);...;F^{e,T}_w(\varphi_i);...;F^{e,T}_w(\varphi_{N^e})]^T$$

$$\mathbf{F}^e_v=[F^e_v(\varphi_1),...,F^e_v(\varphi_i),...,F^e_w(\varphi_{N^e})]^T$$

$$\widehat{\theta}^e=(\mathbf{F}^{e,T}_w\cdot\mathbf{F}^e_w)^{-1}\cdot\mathbf{F}^{e,T}_w\cdot\mathbf{F}^e_v \tag{111}$$

$$\widehat{\theta}_4=\widehat{\theta}^e_1 \tag{112}$$

7. EXPERIMENTAL RESULTS

All Matlab programs developed in (Marin et al., 2005) for time invariant systems have been extended to the time variant systems according the above relations.

Because of limited space, here we include practical results only for a first order nonlinear time variant system characterized by rational dependence on parameters,

$$\dot{y}(t)=\theta_1\cdot y(t)+\frac{\theta_3(t)\cdot u(t)}{\theta_2+y(t)} \tag{113}$$

where

$$\theta_3(t)=\theta_{31}\cdot t+\theta_{30} \tag{114}$$

The vector of unknown parameters to be identified is

$$\theta=[\theta_1\ \ \theta_2\ \ \theta_{31}\ \ \theta_{30}]. \tag{115}$$

This simple model corresponds somehow to the Layer a presented above. It has been utilized testing functions $\varphi(t)$, characterized by a bounded support $T=[t_a,t_b],t_a<t_b$, accomplishing the conditions

$$\varphi(t)=0,\ \forall t\in(-\infty,t_a]\cup[t_b,\infty).$$

$$\varphi(t)=\alpha\cdot\beta(t_a,t_b)\cdot\Psi(t,t_a,t_b)$$

where $\Psi(t,t_a,t_b)=\sin^4[\pi\cdot(t-t_b)/(t_b-t_a)]$

α is a scaling factor and .

$$\beta(t_a,t_b)=1/\int_{t_a}^{t_b}\Psi(t,t_a,t_b),\ \forall t_a,t_b$$

to assure a normalized area.

Figure 1. shows the time variation of unknown coefficients. Two of them are constant $\theta_1=a_1=-0.9$; $\theta_2=a_2=2$, and the third is a sequence of four lines

$$\theta_3(t)=a_3(t)=\theta_{31}\cdot t+\theta_{30}$$

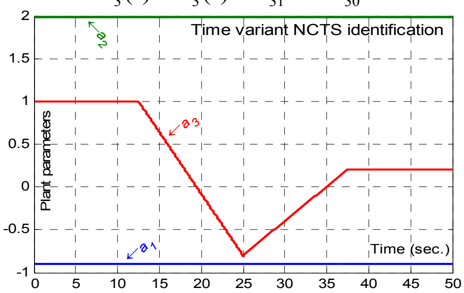

Fig. 1. Time variation of unknown parameters

The measured input-output variables, for the free noise case, are depicted in Figure 2.

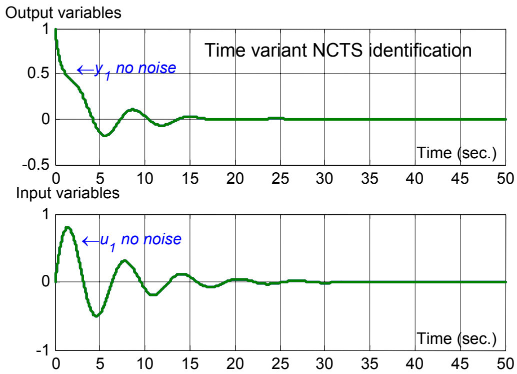

Fig. 2. Time variation of measured input-output variables.

For the first three slopes, the measured variables and identification results are presented as follows:

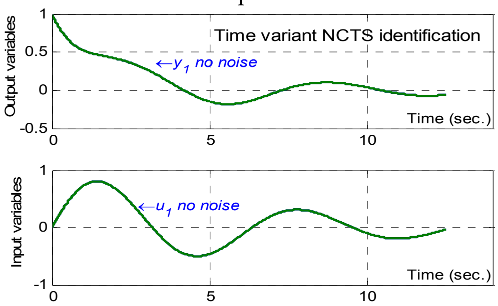

Fig. 3. Time variation of measured input-output variables for the first slope.

Table 1. Results for the first slope

	Real	Identified
θ_2	2.0	2.00000002893786
θ_1	-0.9	-0.90000000953150
θ_{31}	0.0	0.00000000015214
θ_{30}	1.0	1.00000001485449

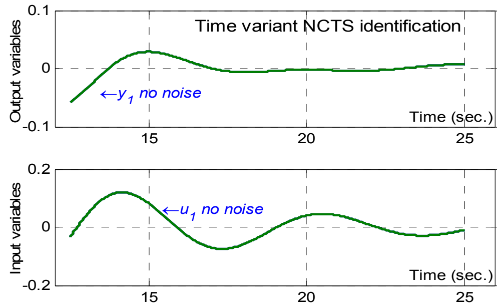

Fig. 4. Time variation of measured input-output variables for the second slope.

Table 2. Results for the second slope

	Real	Identified
θ_2	2.0	2.00000007054610
θ_1	-0.9	-0.90000001125488
θ_{31}	-0.144	-0.14400000531677
θ_{30}	2.8	2.80000010349192

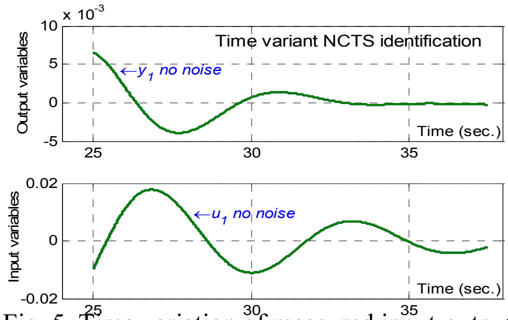

Fig. 5. Time variation of measured input-output variables for the third slope.

Table 3. Results for the third slope

	Real	Identified
θ_2	2.0	1.99999950652750
θ_1	-0.9	-0.89999991867011
θ_{31}	0.08	0.07999998039933
θ_{30}	-2.8	-2.79999931402201

Considering measured output contamined by noise for the input-output pair from Fig.6 and Fig. 7, the identification results for the second slope are presented in Table 4.

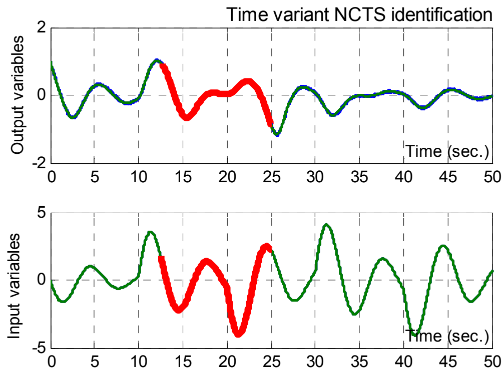

Fig. 6. Time variation of measured variables with noise on output.

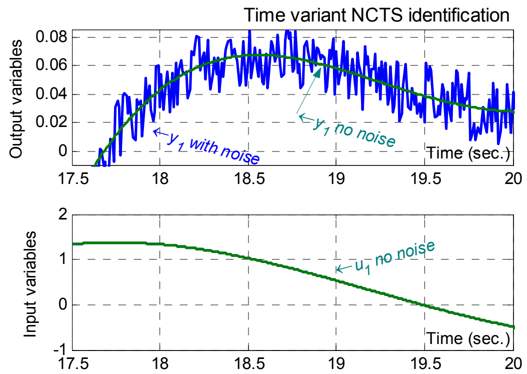

Fig. 7. Zoom on time variation of measured variables with noise on output.

Table 4. Results for the second slop with noise on output measurement.

	Real	Identified
θ_2	2.0	2.01308910556399
θ_1	-0.9	-0.84855617369741
θ_{31}	-0.144	-0.14545065852012
θ_{30}	2.8	2.83139205732140

8. CONCLUSIONS

Through this research has been proved that it is possible to identify all parameters of continuous time nonlinear systems even if they are related to measured variables by rational expressions. This is possible if the identification problem is formulated as a set of interconnected identification problems with linear dependences between parameters and measured variables. The problem of functionals based identification consistency has to be analyzed for a broader class of nonlinear systems.

ACKNOWLEDGMENTS

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