NONLINEAR OSCILLATIONS – FOR AND AGAINST LIAPUNOV FUNCTIONS

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Abstract: Existence of nonlinear oscillations (periodic and almost periodic solutions) for systems with deviated argument and sector restricted (Lurie type) nonlinearities) has been discussed using the Liapunov functional approach as well as the approach of the frequency domain inequality due to V.M. Popov. In this paper a comparison of the computational issues of the two approaches is performed using a standard example supplied by an electrical circuit containing a single lossless transmission line. It is shown that the computational difficulties of the frequency domain method may be overcome when a suitable Liapunov function(al) - here the stored electromagnetic energy - is readily available.

Keywords: nonlinear oscillations, propagation, Liapunov functional, electrical circuits

1. INTRODUCTION AND MOTIVATION

Propagation phenomena are usually considered as describing some dynamics in oscillatory systems with some interconnections which may display space distribution in at least one dimension: propagation is oscillation + waves. Lossless propagation is associated to transmission lines without losses: LC electrical lines, lossless steam, water or gas pipes; a recent reference recalling some classical models is the paper of A. Halanay and Vl. Răsvan(1997).

In electrical engineering applications the propagation problem is closely related to circuit structure consisting of multipoles connected through LC transmission lines. Since we intend to focus namely on such applications we start with an example that is simple enough but nevertheless more complicated than the quasi-benchmark structure with tunnel diode and transmission line introduced by Brayton and Miranker (1964) and discussed by Infante (1971), A. Halanay and Vl. Răsvan (1977, 1997) a.o.

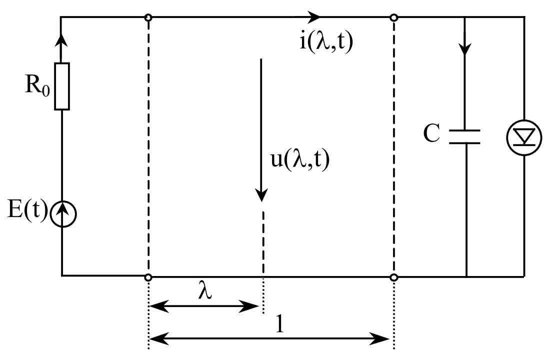

Fig. 1. Nonlinear circuit with a LC line

This circuit contains a nonlinear resistor and this fact complicates the problem of a steady state. Indeed, if the circuit were linear, its steady state would be of the same type as the input signal,

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the source $E(t)$. A harmonic source would generate harmonic currents and voltages which were computable using e.g. the symbolic method due to K.P. Steinmetz. If the source were nonharmonic then several cases would appear: a) a finite superposition of harmonics implying a finite superposition of harmonic steady states for the same frequencies; b) an infinite superposition of harmonics with rationally dependent frequencies i.e. a Fourier series describing a periodic input signal generating a periodic steady state with the same period as the input signal; c) an infinite superposition of harmonics with rationally independent frequencies i.e. a Dirichlet series describing an almost periodic input signal generating an almost periodic steady state.

This reproducibility of the input signal by a linear circuit (or, generally speaking, by a linear dynamical system) is specific. Nevertheless there are nonlinear systems which may preserve some of these properties displaying an almost linear behavior (e.g. Răsvan, 2001). Such an almost linear behavior may be analyzed either by a suitably chosen Liapunov function or by the frequency domain inequality of Popov. The convergence and divergence points of the two methods have been discussed in various contexts (e.g. A. Halanay, 1971, or Vl. Răsvan and S.I. Niculescu, 2002). We want to show here, mainly by computing the above example, how difficult is to decide for one or other approach, thus insisting on application priority.

Consequently the paper is organized as follows: the mathematical model of the circuit is written and a Liapunov functional of the form of the stored electromagnetic energy is introduced. This functional suggests how to choose a genuine Liapunov functional for the associated system of functional differential equations. This functional is discussed in the context of the general Liapunov theory for such equations. Further its Popov-like counterpart is presented in order to make the comparison. Some conclusions are finally presented.

2. THE MATHEMATICAL MODEL AND THE STORED ENERGY

A. The mathematical model of the circuit of fig. 1 is written using the standard development of Kirchhoff laws and the Ohm law with its analogues for electric and magnetic circuit elements

$$\begin{cases} \dfrac{\partial v}{\partial \lambda} = -L\dfrac{\partial i}{\partial t}, \quad \dfrac{\partial i}{\partial \lambda} = -C\dfrac{\partial v}{\partial t}, \\ v(0,t) = v_1(t) \quad , \quad v(1,t) - R_2 i(1,t) = v_2(t) \\ -E(t) - R_1 i^{'}(t) = v_1(t) \ , \ C_2\dfrac{dv_2}{dt} = i(1,t) \\ C_1\dfrac{dv_1}{dt} = -i^{''}(t) \ , \ i^{'''}(t) = -f(v_1(t)) \\ i^{'}(t) + i^{''}(t) + i^{'''}(t) = i(0,t) \end{cases} \tag{1}$$

In the lumped part of the circuit, the standard choice of the state variables corresponds to the voltages across the capacitors and this choice is consistent with the fact that the nonlinear resistor is voltage controlled. We may eliminate the current variables $i^{'}, i^{''}, i^{'''}$ which are linearly dependent of the state variables to obtain the following model

$$\begin{cases} \dfrac{\partial v}{\partial \lambda} = -L\dfrac{\partial i}{\partial t}, \quad \dfrac{\partial i}{\partial \lambda} = -C\dfrac{\partial v}{\partial t}, \\ v(0,t) = v_1(t) \quad , \quad v(1,t) - R_2 i(1,t) = v_2(t) \\ R_1 C_1\dfrac{dv_1}{dt} = -v_1 - R_1 f(v_1) - R_1 i(0,t) - E(t) \\ R_2 C_2\dfrac{dv_2}{dt} = R_2 i(1,t) \end{cases} \tag{2}$$

(the multiplication with R_i in the last equations aimed to point out the time constants $R_i C_i$)

Let us comment on this model. We have here a simple hyperbolic system of partial differential equations with standard boundary conditions which are in "internal feedback" with some ordinary differential equations at both boundaries. Such systems are quite well known. the reader is sent to author's book (1975), also to the survey of A. Halanay and the author (1997) as well as to the more recent book of Niculescu (2001). Without additional details, let us mention that a one-to-one correspondence may be established between thew solutions of (2) and the solutions of the following system of functional differential equations

$$\begin{cases} R_1 C_1\dfrac{dv_1}{dt} = -(1 + R_1\sqrt{C/L})v_1(t) + \\ +2R_1\sqrt{C/L}\eta_2(t - \sqrt{LC}) - R_1 f(v_1(t)) - E(t) \\ (1 + R_2\sqrt{C/L})R_2 C_2\dfrac{dv_2}{dt} = -R_2\sqrt{C/L}(v_2(t) - \\ -2\eta_1(t - \sqrt{LC})) \\ \eta_1(t) = v_1(t) - \eta_2(t - \sqrt{LC}) \\ \eta_2(t) = \dfrac{1}{1 + R_2\sqrt{C/L}}v_2(t) - \\ -\dfrac{1 - R_2\sqrt{C/L}}{1 + R_2\sqrt{C/L}}\eta_1(t - \sqrt{LC}) \end{cases} \tag{3}$$

This correspondence is obtained by introducing the functions

$$\begin{cases} \eta_1(t) = \dfrac{1}{2}(v(0,t) + \sqrt{L/C}i(0,t)) \\ \eta_2(t) = \dfrac{1}{2}(v(1,t) + \sqrt{L/C}i(1,t)) \end{cases} \tag{4}$$

which satisfy (3) and represent the solutions of (2) as

$$\begin{cases} v(\lambda,t) = \eta_1(t-\lambda\sqrt{LC}) + \eta_2(t-(1-\lambda)\sqrt{LC}) \\ i(\lambda,t) = \sqrt{C/L}\left(\eta_1(t-\lambda\sqrt{LC}) - \right. \\ \left. - \eta_2(t-(1-\lambda)\sqrt{LC})\right) \end{cases} \tag{5}$$

B. We associate now to system (2) the *stored electromagnetic energy*, taking into account that it is stored in the lumped capacitors at the boundaries of the line and in the distributed capacitors and inductors of the line

$$\begin{cases} \mathcal{E}(v_1,v_2,i,v) = \dfrac{1}{2}[(C_1v_1^2 + C_2v_2^2) + \\ + \displaystyle\int_0^1 (Li^2(\lambda,t) + Cv^2(\lambda,t))d\lambda] \end{cases} \tag{6}$$

which clearly looks like a functional on some state space; at its turn this state space appears as a Cartesian product of some finite dimensional space (here $\mathbb{R}^2$) and some function space that accounts for the distributed parameters along the LC transmission line.

Let $(v_1(t), v_2(t), i(\lambda,t), v(\lambda,t))$ be a solution of (2) - assumed to exist - and write down the stored energy along this solution

$$\mathcal{E}^\star(t) \equiv \mathcal{E}(v_1(t), v_2(t), i(\lambda,t), v(\lambda,t)) \tag{7}$$

Differentiating $\mathcal{E}^\star(t)$ - formally, since nothing is known about solution smoothness - we obtain

$$\begin{cases} \dfrac{d\mathcal{E}^\star}{dt} = C_1v_1(t)\dot{v}_1(t) + C_2v_2(t)\dot{v}_2(t) + \\ + \displaystyle\int_0^1 [Li(\lambda,t)\frac{\partial i}{\partial t}(\lambda,t) + Cv(\lambda,t)\frac{\partial v}{\partial t}(\lambda,t)]d\lambda \end{cases}$$

and, taking into account (2)

$$\begin{cases} \dfrac{d\mathcal{E}^\star}{dt} = -v_1(t)(\dfrac{v_1(t)}{R_1} + f(v_1(t)) - \\ -R_2i_1^2(1,t) - \dfrac{v_1(t)}{R_1}E(t) \end{cases} \tag{8}$$

For the case of the "free" circuit i.e. with $E(t) \equiv 0$ and moving under the initial conditions only, we obtain $d\mathcal{E}^\star/dt \leq 0$ i.e. *a non-increasing stored energy* because of the dissipation in the linear lumped resistors R_1, R_2 as well as in the linear lumped voltage controlled resistor $f(v_1)$. This will ensure in any case Liapunov stability *in the sense of the norm induced by the energy function itself*

$$\begin{aligned} &\mathcal{E}(v_1(t), v_2(t), i(\cdot,t), v(\cdot,t)) \leq \\ &\leq \mathcal{E}(v_1(0), v_2(0), i_0(\cdot), v_0(\cdot)) \end{aligned} \tag{9}$$

even if a certain "negative resistance" is allowed by the sector (Lurie type condition)

$$f(\sigma)/\sigma \geq -\frac{1}{R_1} \tag{10}$$

On the other hand it is known that even a lossless line has an equivalent impedance that may modify dissipation and this impedance does not appear at all in the above discussion. Overcoming the drawbacks of a Liapunov function(al) is not an easy task. Our approach is to try to improve it using Popov theory which asserts that a frequency domain inequality is equivalent to the existence of the most general Liapunov function(al) within a certain class.

Since there are also other obscure points in the above construction, it appears as rational to use the associated system (3) and the representation formulae (5). We deduce that $\mathcal{E}(v_1, v_2, i(\cdot), v(\cdot))$ defined by (6) may be expressed using (5)

$$\begin{aligned} &V(v_1,v_2,\eta_1(\cdot),\eta_2(\cdot)) = \frac{1}{2}(C_1v_1^2 + C_2v_2^2) + \\ &+\sqrt{C/L}\int_{-\sqrt{LC}}^{0} (\eta_1^2(\theta) + \eta_2^2(\theta))d\theta \end{aligned} \tag{11}$$

that is a functional of the simplest diagonal form. This will be a good suggestion for the choice of a suitable Liapunov function for the system (3) of functional differential equations.

3. ALMOST LINEAR BEHAVIOR

Almost linear behavior for nonlinear systems has been defined in e.g. (Barbălat and Halanay, 1974; Răsvan, 2001). It means fulfilment of two basic properties: a) the system has a single globally asymptotically (possibly exponentially) stable equilibrium; b) if the system is excited (forced) by a bounded "oscillatory" input signal (constant, periodic, almost periodic) then it has a unique bounded on $\mathbb{R}$ (the whole real axis - the "time") which is of the same type ("shape") as the excitation (i.e. constant, periodic with the same period or almost periodic) and moreover, it is exponentially stable.

We shall consider this major qualitative property for a class of F(unctional) D(ifferential)

E(quations) that incorporates the example analyzed in the previous sections, namely the system

$$\begin{cases} \dot{x}_1(t) = A_0(t)x_1(t) + A_1(t)x_2(t-\tau)- \\ -b_1(t)\phi(t,\sigma(t)) + f_1(t) \\ x_2(t) = A_2(t)x_1(t) + A_3(t)x_2(t-\tau)- \\ -b_2(t)\phi(t,\sigma(t)) + f_2(t) \\ \sigma = c^*(t)x_1 \end{cases} \tag{12}$$

with T-periodic coefficients $A_i(t)$, $b_j(t)$, $i = 0,1,2,3$, $j = 1,2$; also $\phi(\cdot,\sigma)$ is T-periodic but the delay $\tau > 0$ is nevertheless assumed constant. Also the following globally Lipschitz condition is fulfilled for ϕ

$$\begin{aligned} &0 \leq (\phi(t,\sigma_1) - \phi(t,\sigma_2))/(\sigma_1 - \sigma_2) \leq \\ &\leq \bar{\phi} < +\infty \end{aligned} \tag{13}$$

for all $\sigma_1 \neq \sigma_2$ and all $t \in [0,T)$. before discussing the main results some comments are necessary since they send to paper's title. This title is partly reproducing that of (Halanay, 1971) - a "state of the art" account of the genuine competition existing between the Liapunov method and that of the frequency domain inequalities in the field of stability and stable oscillations for systems with sector restricted nonlinearity. Interesting comments on this subject may be found e.g. in (*op. cit.*) also in (Niculescu and Răsvan, 2002). here we shall focus mainly on some applications to system (12).

A. The first (chronologically speaking) result is based on Popov frequency domain inequality and reads as follows

Theorem 1. Consider the nonlinear system (12) under the following assumptions: i) the coefficients of the linear part i.e.A_i, $i = 0,1,2,3$, b_i, $i = 1,2$ and c are time invariant;ii) the linear part is exponentially stable i.e. the linear system with constant coefficients

$$\begin{cases} \dot{x}_1(t) = A_0x_1(t) + A_1x_2(t-\tau) \\ x_2(t) = A_2x_1(t) + A_3x_2(t-\tau) \end{cases} \tag{14}$$

has the roots of the characteristic equation

$$\det\begin{pmatrix} \lambda I - A_0 & -A_1e^{-\lambda\tau} \\ -A_2 & I - A_3e^{-\lambda\tau} \end{pmatrix} = 0 \tag{15}$$

in the open L(eft) H(alf) P(lane) $\mathbb{C}^-$ (in a necessary way this will require the eigenvalues of A_3 to be located inside the unit disk $\mathbb{D}_1 \subset \mathbb{C}$); iii) the nonlinear function$\phi(t,\sigma)$ satisfies (13) for all $\sigma_1 \neq \sigma_2$ and real t; iv) the circle-like frequency domain inequality

$$\frac{1}{\bar{\phi}} + \Re e H(\jmath\omega) > 0, \tag{16}$$

where, as usual,

$$H(s) = \begin{pmatrix} c^* & 0 \end{pmatrix} \begin{pmatrix} sI - A_0 & -A_1e^{-s\tau} \\ -A_2 & I - A_3e^{-s\tau} \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

holds for all $\omega > 0$; v) $|f_1(t)| + |f_2(t)| < M$. Then there exists a bounded on the entire real axis solution of (12) which is globally exponentially stable. If f_1, f_2 and $\phi(\cdot,\sigma)$ are periodic with rationally dependent periods, this solution is periodic. If these periods are rationally independent or at least one of these functions is almost periodic, then the solution is almost periodic.

We send the reader to the corresponding reference (Halanay and Răsvan, 1977) and focus on the application of the theorem to the circuit of fig. 1 i.e. to system (3). Nevertheless we perform first a "sector rotation" for the nonlinearity, replacing the first equation of (3) by the following

$$\begin{aligned} &R_1C_1\frac{dv_1}{dt} = -av_1(t)+ \\ &+2R_1\sqrt{C/L}\eta_2(t-\sqrt{LC}) - \varphi(v_1(t)) - E(t) \end{aligned} \tag{17}$$

where

$$\varphi(\sigma) := (1 + R_1\sqrt{C/L} - a)v_1 + R_1f(v_1) \tag{18}$$

with $a > 0$ chosen appropriately. We check the assumptions of Theorem 1. The characteristic equation (15) is here (after some straightforward manipulation based on the well known formulae of Schur)

$$\begin{aligned} &\delta(\lambda) \equiv (T_1\lambda + a)(T_2\lambda + R_2\sqrt{C/L})- \\ &-(T_1\lambda + a - 2R_1\sqrt{C/L})\times \\ &\times(\rho T_2\lambda - R_2\sqrt{C/L})e^{-\lambda\tau} = 0 \end{aligned} \tag{19}$$

where we denoted

$$\begin{aligned} &T_1 = R_1C_1,\ T_2 = (1 + R_2\sqrt{C/L})C_2\sqrt{C/L}, \\ &\rho = (1 - R_2\sqrt{C/L})((1 + R_2\sqrt{C/L})^{-1}, \\ &\tau = 2\sqrt{LC} \end{aligned} \tag{20}$$

Checking the necessary (Stodola-like conditions) for the $\mathbb{C}^-$ root location of (19) - see the memoir of Čebotarev and Meĭman (1949) - we obtain $a > R_1\sqrt{C/L}$. Since we are interested in the largest sector of the nonlinear function for the almost linear behavior, take $a = R_1\sqrt{C/L}$ (for a while). Then (19) becomes

$$\begin{aligned} &(T_1\lambda + R_1\sqrt{C/L})(T_2\lambda + R_2\sqrt{C/L})- \\ &-(T_1\lambda - R_1\sqrt{C/L})(\rho T_2\lambda - R_2\sqrt{C/L})e^{-\lambda\tau} = 0 \end{aligned} \tag{21}$$

which has $\lambda = 0$ as a root - a critical case. Since existence of globally exponentially stable forced oscillations is not solved in critical cases even in the case without delay and Theorem 1 assumptions require all roots in $\mathbb{C}^-$ we shall thus take $a = R_1\sqrt{C/L}+\varepsilon$ with $\varepsilon > 0$ arbitrarily small but well delimited from 0. Write now (19) as

$$\begin{aligned}&(T_1\lambda + R_1\sqrt{C/L} + \varepsilon)(T_2\lambda + R_2\sqrt{C/L})\times\\ &\times\left(1 - \frac{T_1\lambda - R_1\sqrt{C/L}+\varepsilon}{T_1\lambda + R_1\sqrt{C/L}+\varepsilon}\times\right.\\ &\left.\times\frac{\rho T_2\lambda - R_2\sqrt{C/L}}{T_2\lambda + R_2\sqrt{C/L}}e^{-\lambda\tau}\right) = 0\end{aligned} \tag{22}$$

and we may show that for $\bar{\alpha} > 0$ sufficiently small equation (19) cannot have roots of the form $-\alpha + \jmath\omega$ provided $\alpha < \bar{\alpha}$ i.e. all roots of (19) satisfy $\Re e(\lambda) \leq -\bar{\alpha}$. Assume first that neither $-(\varepsilon + R_1\sqrt{C/L})/T_1$ nor $-(R_2\sqrt{C/L})/T_2$ are roots of (19); in this case all roots of (19) - or (22) - are the roots of

$$\begin{aligned}&\left(1 - \frac{T_1\lambda - R_1\sqrt{C/L}+\varepsilon}{T_1\lambda + R_1\sqrt{C/L}+\varepsilon}\times\right.\\ &\left.\times\frac{\rho T_2\lambda - R_2\sqrt{C/L}}{T_2\lambda + R_2\sqrt{C/L}}e^{-\lambda\tau}\right) = 0\end{aligned} \tag{23}$$

It is easy to show first that (23) cannot be 0 for $\lambda = \jmath\omega$ i.e. on the imaginary axis $\jmath\mathbb{R}$. Indeed we have

$$\begin{aligned}&\frac{(R_1\sqrt{C/L}-\varepsilon)^2 + \omega^2T_1^2}{(R_1\sqrt{C/L}+\varepsilon)^2 + \omega^2T_1^2}\times\\ &\times\frac{R_2^2(C/L) + \rho^2T_2^2\omega^2}{R_2^2(C/L) + T_2^2\omega^2} < 1\end{aligned} \tag{24}$$

since each ration in (24) is less than 1 - the first because $\varepsilon > 0$ and the second because $\rho^2 < 1$;*the first condition has been assumed above* and the second is obviously fulfilled. remark also that for $\omega \to \infty$ the second ratio equals $\rho^2 < 1$ while the first one equals 1 hence their product is still less than 1.

This means that the modulus of the expression subtracted from 1 in (23) never reaches 1 on $\jmath\mathbb{R}$ and the maximum modulus principle says that in $\mathbb{C}^-$ this is also true. Since we want to avoid root accumulation in the neighborhood of $\jmath\mathbb{R}$, consider now $\lambda = -\alpha + \jmath\omega$ with $\alpha > 0$. The analyzed modulus is now

$$\begin{aligned}\mu(\omega) = &\frac{(R_1\sqrt{C/L}-\varepsilon+\alpha T_1)^2 + \omega^2T_1^2}{(R_1\sqrt{C/L}+\varepsilon-\alpha T_1)^2 + \omega^2T_1^2}\times\\ &\times\frac{(R_2\sqrt{C/L}+\alpha\rho T_2)^2 + \rho^2T_2^2\omega^2}{(R_2\sqrt{C/L}-\alpha T_2)^2 + T_2^2\omega^2}e^{\alpha\tau}\end{aligned}$$

and from continuity with respect to $\alpha > 0$ small enough we deduce that $\mu(\omega) < 1$ for all $\omega \in \mathbb{R}$ provided $\rho^2 < 1$. If either $-(\varepsilon + R_1\sqrt{C/L})/T_1$ or $-(R_2\sqrt{C/L})/T_2$ is a root of (19) then a factor with a root in $\mathbb{C}^-$ is simplified and we repeat the above construction for a single modulus ratio. It is obvious that both $-(\varepsilon + R_1\sqrt{C/L})/T_1$ and $-(R_2\sqrt{C/L})/T_2$ cannot be roots of (19). In this way *we checked fulfilment of the second assumption of* Theorem 1 *by system* (3) *with the rotated nonlinearity*: its linear part is exponentially stable.

The transfer function of system's linear part reads as

$$H(s) = \nu(s)/\delta(s) \tag{25}$$

where the denominator $\delta(s)$ is defined in (19) being exactly system's characteristic equation and the numerator $\nu(s)$ is written below

$$\nu(s) = T_2s + R_2\sqrt{C/L} - (\rho T_2s - R_2\sqrt{C/L})e^{-\tau s}$$

and it may be checked *via* a lengthy computation that $\Re eH(\jmath\omega) \geq 0$, $\forall\omega \in \mathbb{R}$.

In this way the assumptions of Theorem 1 are all checked; its application will give existence and exponential stability of forced (almost) periodic oscillations.

B. As already mentioned, checking of (16) may be somehow lengthy even on some low order system, as the considered example. On the contrary, a suitably chosen Liapunov functional, suggested by the energy function - which is well known in the case of the electrical circuits - transformed *via* the representation formulae (5) - may give equivalent results with less computational effort. We state here the Liapunov functional result

Theorem 2. Consider system (12) and assume that i), ii) and v) of Theorem 1 hold. Assume also that: iv') there exist positive definite matrices P and R such that the following matrix inequalities hold

$$\begin{aligned}&\begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{12}^* & H_{22} & H_{23} \\ H_{13}^* & H_{23}^* & H_{33}\end{pmatrix} \leq 0\ ,\\ &\begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22}\end{pmatrix} < 0\end{aligned} \tag{26}$$

where we denoted

$$H_{11} = A_0^* P + P A_0 + A_2^* R A_2,$$
$$H_{12} = P A_1 + A_2^* R A_3$$
$$H_{13} = -(P b_1 + A_2^* R b_2) + \frac{1}{2} c, \quad (27)$$
$$H_{22} = A_3^* R A_3 - R$$
$$H_{23} = -A_3^* R b_2, \; H_{33} = b_2^* R b_2 - \frac{1}{\bar{\phi}}$$

Then the conclusion of Theorem 1 holds.

The proof of this theorem may be found in (Niculescu and Răsvan, 2002); we mention nevertheless that it is mainly based on the properties of the Liapunov functional

$$V(x_1, x_2(\cdot)) = x_1^* P x_1 + \int_{-\tau}^{0} x_2^*(\theta) R x_2(\theta) d\theta \quad (28)$$

where P and R are those of Theorem 2.

We apply further this result to our example. First we rewrite its equations according to the change of the nonlinear function (18) with the choice suggested by the Stodola-like conditions, $a = R_1\sqrt{C/L}$

$$\begin{cases} R_1 C_1 \dfrac{dv_1}{dt} = 2R_1\sqrt{C/L}\eta_2(t-\sqrt{LC}) - \\ -R_1\varphi(v_1(t)) - E(t) \\ (1+R_2\sqrt{C/L})R_2C_2\dfrac{dv_2}{dt} = -R_2\sqrt{C/L}(v_2(t) - \\ -2\eta_1(t-\sqrt{LC})) \\ \eta_1(t) = v_1(t) - \eta_2(t-\sqrt{LC}) \\ \eta_2(t) = \dfrac{1}{1+R_2\sqrt{C/L}} v_2(t) - \\ -\dfrac{1-R_2\sqrt{C/L}}{1+R_2\sqrt{C/L}}\eta_1(t-\sqrt{LC}) \end{cases} \quad (29)$$

where $\varphi(v_1) = (1+R_1\sqrt{C/L})v_1 + R_1 f(v_1)$.

The Liapunov functional is suggested by (11) but we shall introduce also some free parameters for "optimization" (i.e. some additional available choice)

$$V(v_1, v_2, \eta_1(\cdot), \eta_2(\cdot)) = \frac{1}{2}[R_1C_1v_1^2 + a_1(1+R_1\sqrt{C/L})R_2C_2v_2^2] + \int_{-\sqrt{LC}}^{0} \left(a_2\eta_1^2(\theta) + a_3(1+R_1\sqrt{C/L})^2\eta_2^2(\theta)\right) d\theta \quad (30)$$

with $a_i > 0$, $i = 1, 2, 3$ subject to choice. Since all physical parameters are positive, the functional is clearly positive definite on $\mathbb{R}^2 \times L^2(-\sqrt{LC}, 0; \mathbb{R}^2)$. We compute its derivative along the solutions of (29) i.e. we differentiate $V^\star(t) = V(v_1(t), v_2(t), \eta_1(t+\cdot), \eta_2(t+\cdot))$ and take into account (29) to obtain

$$\begin{aligned} & W_t(v_1, v_2, \eta_1(\cdot), \eta_2(\cdot)) = -v_1\varphi(v_1) + a_2v_1^2 + \\ & +2(R_1\sqrt{C/L} - a_2)v_1\eta_2(-\sqrt{LC}) - \\ & -(a_3(1+R_1\sqrt{C/L})^2 - a_2)\eta_2^2(-\sqrt{LC}) - \\ & -(a_1R_2\sqrt{C/L} - a_3)v_2^2 + \\ & +2(a_1R_2\sqrt{C/L} - a_3)v_2\eta_1(-\sqrt{LC}) - \\ & -(a_2 - a_3(1-R_2\sqrt{C/L})\eta_1^2(-\sqrt{LC}) - v_1E(t) \end{aligned} \quad (31)$$

In (31) we see two decoupled second order quadratic forms that have to be negative definite by a suitable choice of $a_i > 0$ and of some $\delta_0 > 0$ such that

$$v_1\varphi(v_1) \geq (a_2 + \delta_0)v_1^2 \quad (32)$$

Since we search for the largest possible sector of the nonlinearity, $a_2 > 0$ has to be as small as allows the choice for the negative definiteness of the two quadratic forms of (31). Some simple but not quite straightforward computation shows that this choice is given by the inequalities below (deduced from two independent sets of Sylvester conditions)

$$\begin{cases} \dfrac{R_1\sqrt{C/L}}{(1+R_2\sqrt{C/L})^2} < \dfrac{a_3}{a_1} + \\ +\dfrac{1}{4(1+R_2\sqrt{C/L})^2} \cdot \dfrac{\delta_0}{a_1} < \\ < R_2\sqrt{C/L} + \dfrac{1}{4(1+R_2\sqrt{C/L})^2} \cdot \dfrac{\delta_0}{a_1}; \\ \dfrac{a_2}{a_1} > R_2\sqrt{C/L}\psi(R_2\sqrt{C/L}\dfrac{a_1}{a_3} \end{cases} \quad (33)$$

where

$$\psi(x) = 1 + \frac{R_2\sqrt{C/L}}{x} + \frac{(R_2\sqrt{C/L})^2}{x(x-1)} \quad (34)$$

with $x > 1$. It is easy to see that no generality is affected by choosing $a_1 = 1$. Since $\psi(x) > 1$ and is monotonically decreasing for $x > 1$ we need a_3 as small as possible to obtain the RHS of the second inequality of (33) as small as possible; this will lead to the smallest possible a_2 hence to the largest possible sector.

4. CONCLUSIONS AND PERSPECTIVE

We have given in this paper several points of contact and results connected to the standard by now dialectics of the two methods applied in the qualitative analysis of the systems with sector restricted nonlinearities: the Liapunov function(al) method and the Popov frequency domain inequality method. These methods have been shown since more than four decades to be equivalent *via* positivity theory (Yakubovich-Kalman-Popov lemma) which has been extended some three decades ago to Hilbert spaces being thus applicable to time delay and propagation systems. In practice the things are different and this may be seen on our example. The ultimate truth of the positivity theory is that the frequency domain inequality (which depends in a unique way of system's structure and parameters - it is "mechanically" obtained) is equivalent to existence of the most general Liapunov function(al) with the structure "quadratic form (on the Hilbert state space) plus the integral of the nonlinear function"

$$V(x) = (x, Hx) + \beta \int_0^{(c,x)} \varphi(\lambda) d\lambda \qquad (35)$$

where x denotes the state on the Hilbert space, $(\ ,\ )$ is the scalar product on that Hilbert space and $H = H^*$ is a Hermitian operator which is subject to some Linear Operator Inequalities of Riccati type. Solution of these inequalities is rather difficult. We restricted ourselves to a special case leading to Linear Matrix Inequalities which provide delay independent conditions of almost linear qualitative behavior. Moreover, in such applications as electric circuits, where a Liapunov functional is naturally associated, as an energy-like functional, the results may be quite complete, with a minimum of computation (like in our example). Besides computational effort, maximal performance is also a goal of the approach. At the nonlinearity level the evaluation is given by the almost linear behavior sector i.e. by the answer to the old problem of Aizerman, stated in the 40ies but still of interest (see the book of Egorov, 1998 and, for delay systems, the quite recent paper of the author, 2004). The sharpest results are given by the Liapunov function(al), the most general (i.e. invariant with respect to the state space choice) by the frequency domain inequality. The computational effort may give priority to any of the two methods according to the specific application. hence, as in 1971 (A. Halanay) or in 2002 (Vl. Răsvan and S.I. Niculescu) the question still stays: for or against Liapunov functions?

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