GLOBAL EXPONENTIAL STABILITY OF HOPFIELD-TYPE NEURAL NETWORKS WITH TIME DELAYS

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Abstract: The paper is concerned with the improvement of sufficient conditions for the exponential stability of Hopfield-type neural networks displaying interaction delays. The results are based on a method obtained in our previous work that combines an idea suggested by Malkin for studying the absolute stability of a nonlinear system via their linearisations and a procedure proposed by Kharitonov for construction of an “exact” Liapunov-Krasovskii functional used in the analysis of uncertain linear time delay systems. Since the Liapunov function method give only sufficient conditions for stability, the improvement of these criteria is obviously necessary. These less conservative conditions are suitable for the implementation of recurrent neural networks.

Keywords: Hopfield neural network, time delays, asymptotic stability, robust stability.

1. INTRODUCTION

Neural networks are structures that possess “emergent computational capabilities”, that is they consist of *interconnected simple computational devices* to which interconnections confer increased computational power, property which cannot be inferred from the propeties of an individual element. The field of neural networks deals with two types of networks (Bose and Liang, 1996): feedforward and recurrent neural networks. Feedforward networks implement mappings from the input pattern space to the output space. Once the interconnection weights obtained by a training process are fixed, the neurons are totally determined by inputs, independent of their past states. Feedforward networks do not display any dynamics. For more complicated information processing one needs recurrent neural networks (RNN). Due to theirs cyclic interconnections RNN are dynamical systems with very rich spatial and temporal behaviours: stable and unstable fixed points, limit cycles and chaotic behaviour. These behaviours make them suitable for modeling certain cognitive functions such as associative memory, unsupervised learning, self-organizing maps and temporal reasoning.

The mathematical models of recurrent neural networks arise both from the modeling of some behaviours of biological structures or from the necessity of Artificial Inteligence to consider some structures which solve certain tasks. None of these two cases has as primarly aim stability aspects and a “good” qualitative behaviour. On the other hand, these properties are necessary and therefore important for the network to achieve its purpose.

Hopfield-type neural networks are recurrent networks consisting of one layer of totally interconnected neurons. Symmetric Hopfield networks model associative memories. In this case the stable fixed points in state space store stationary patterns which has to be retrieved from partial or distorted information. Asymmetric Hopfield networks with limit cycle attractors can be used for associative memories of temporal sequences of patterns and also as pattern sequence generators. In both cases the stability of the equilibria is equally important. On the other hand, the existence of reacting time delays may introduce oscillations or may lead to instability of the network.

Since the mathematical tools for checking the stability properties gives only sufficient conditions, these criteria deserve a permanent improvement. This means sharper and less conservative conditions, which can be relatively easy to check on a given network.

2. THE MATHEMATICAL MODEL AND PROBLEM STATEMENT

We consider the standard equations for the Hopfield network affected by time delays at the interconnection level (see Gopalsami and He, 1994; van den Driessche and Zou, 1998):

$$\dot{x}_i(t) = -a_i x_i(t) - \sum_{j=1}^{n} c_{ij} g_j\left(x_j(t-\tau_j)\right) + I_i\,, \quad i = \overline{1,n} \tag{1}$$

where:

- x_i - the state of the neuron i;
- c_{ij} - the synaptic weight between neurons i and j
- $g(\cdot)$ - the nonlinear activation function;
- a_i - a positive parameter;
- I_i - the bias for the neuron i;
- j - the index for the neurons connected with the neuron i;
- τ_j - the time-lag associated to each interconnection from the neuron j to the neuron i; $j = \overline{1,n}$.

The nonlinearities for Hopfield networks are of the sigmoidal type; some examples of such functions are the following:

$$\begin{gathered} g(\sigma) = \tanh(\sigma) = \frac{e^{\sigma} - e^{-\sigma}}{e^{\sigma} + e^{-\sigma}} \\ g(\sigma) = \frac{1-e^{-\sigma}}{1+e^{-\sigma}} = \tanh\left(\frac{\sigma}{2}\right) \\ g(\sigma) = \frac{2}{\pi}\tan^{-1}\left(\frac{2\sigma}{\pi}\right) \end{gathered} \tag{2}$$

All sigmoidal functions are bounded: more specific their range is [-1, 1]. Also these functions are monotonically increasing and globally Lipschitzian. This means they satisfy the inequalities:

$$0 \le \frac{g(\sigma_1) - g(\sigma_2)}{\sigma_1 - \sigma_2} \le L \tag{3}$$

For instance, the Lipschitz constant for the functions defined in (2) is 1, 1/2, $4/\pi^2$ respectively.

The goal of the paper is to obtain sufficient conditions for the asymptotic stability of the Hopfield-type network with time delay feedback described by equations (1). Due to their properties, the sigmoidal functions belong to the class of the sector restricted (Lurie type) nonlinearities what sends to the absolute stability problem. In fact, this problem is a robustness problem with respect to an entire class of nonlinear functions.

Since the simplest Liapunov function(al) in this case is the quadratic one, we focus on such function(al)s. In the linear case a quadratic Liapunov functional may provide necessary and sufficient conditions for exponential stability, but in the time delay case the sharpest most general quadratic Liapunov function (as suggested by the papers of Datko and Infante with Castelan - their exact references are to be found in Gu, Kharitonov and Chen, 2003) is rather difficult to manipulate. On the other hand, the simplified versions which are currently used (including our reference D. Danciu and Vl. Rasvan, 2000, 2001a, 2001b, 2005, D. Danciu, 2002, 2004) deserve improvement.

In order to achieve our aim one uses a method proposed in our preview work (Danciu, 2004; Danciu and Rasvan, 2005) which combines an idea suggested by Malkin for studying the absolute stability of a nonlinear system via their linearisations and a procedure proposed by Kharitonov for construction of an “exact” Liapunov-Krasovskii functional of the quadratic form used for the analysis of uncertain linear time delay systems.

3. MAIN RESULT

Let x^* be the equilibrium point for the system (1). Without loss of generality, using the change of coordinates $z_i = x_i - x_i^*$, one can shift the equilibrium to the origin, so that system (1) may be written into the form:

$$\dot{z}_i(t) = -a_i z_i(t) + \sum_{j=1}^{n} c_{ij} f_j\left(z_j(t-\tau_j)\right), \quad i = \overline{1,n} \tag{4}$$

where

$$f_i(z_i) = g_i(z_i + x_i^*) - g_i(x_i^*)\,,\ \forall i\,. \tag{5}$$

Denoting

$$A_0 = diag\left(-a_i\right)_1^n \qquad C_j = \begin{pmatrix} 0 & \dots & 0 & c_{1j} & 0 & \dots & 0 \\ 0 & \dots & 0 & c_{2j} & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & c_{nj} & 0 & \dots & 0 \end{pmatrix} \tag{6}$$

one obtain the matriceal form of (4):

$$\dot{z}(t) = A_0 z(t) + \sum_{j=1}^{n} C_j diag\left(f_p\left(z_p\left(t-\tau_j\right)\right)\right) \qquad (7)$$

with the initial condition $z_i(\theta) = \varphi(\theta)$, for $\theta \in [-\tau, 0]$, where $\tau = \max_j \tau_j$ and $\varphi \in \boldsymbol{C}([-\tau, 0], \boldsymbol{R}^n)$.

Regarding the nonlinearities f_i, if we take into account their definition (5) and the properties of functions g_i, we shall obtain the following sector-type restrictions

$$0 \le \frac{f_i(\sigma)}{\sigma} \le L_i \qquad (8)$$

what send us to an absolute stability problem for the nonlinear system (7).

Following our method (Danciu and Rasvan, 2005), at this point we apply the idea suggested by Malkin. We assume that there exist $k_j > 0$, $j = \overline{1, n}$ such that for $f_j(z_j) = k_j z_j$ the linearized system

$$\dot{z}_i(t) = -a_i z_i(t) + \sum_{j=1}^{n} c_{ij} k_j z_j(t-\tau_j)\ ,\ i = \overline{1, n} \qquad (9)$$

with its matriceal form

$$\dot{z}(t) = A_0 z(t) + \sum_{j=1}^{n} A_j z\left(t-\tau_j\right) \qquad (10)$$

where

$$A_j = C_j \cdot diag(k_i)_1^n \qquad (11)$$

is exponentially stable and consider the perturbed system

$$\dot{y}(t) = A_0 y(t) + \sum_{j=1}^{n} C_j diag\left(k_p + b_p\right) y\left(t-\tau_j\right) \qquad (12)$$

with $b_i \in \left(-\underline{k_i}, \overline{k_i}\right)$, $\forall i = \overline{1, n}$. Denoting $\Delta_j = C_j diag(b_p)$, system (12) may be re-written as

$$\dot{y}(t) = A_0 y(t) + \sum_{j=1}^{n} \left(A_j + \Delta_j\right) y\left(t-\tau_j\right). \qquad (13)$$

The robustness problem with respect to these linear uncertainties is to find conditions such that the perturbed system (13) remains exponentially stable for all $b_i \in \left(-\underline{k_i}, \overline{k_i}\right)$, $\forall i = \overline{1, n}$. This is ensured by the Kharitonov-like approach (Kharitonov and Zhabko, 2001; Gu, Kharitonov and Chen, 2003) for the analysis of uncertain linear time delay systems.

Now, for given $\sigma \neq 0$ we may always find some $b_j(\sigma)$ from

$$b_j(\sigma) = \frac{f_j(\sigma)}{\sigma} - k_j \qquad (14)$$

and consider the system

$$\dot{y}(t) = A_0 y(t) + \sum_{j=1}^{n} \left(A_j + \Delta_j(\sigma)\right) y\left(t-\tau_j\right) \qquad (15)$$

with $\Delta_j(y) := C_j diag\left(b_p\left(y_p\left(t-\tau_j\right)\right)\right)$. The idea of this substitution belongs to Malkin. If we succeed in showing stability preservation for $b_i \in \left(-\underline{k_i}, \overline{k_i}\right)$, $\forall i = \overline{1, n}$, then we have obtained absolute (robust) stability for the nonlinear functions satisfying

$$0 \le k_i - \underline{k_i} < \frac{f_i(\sigma)}{\sigma} < k_i + \overline{k_i}\ . \qquad (16)$$

Since linear system (10) is assumed exponentially stable, there exists a positive definite Liapunov functional of the quadratic type whose derivative along the solutions of the system is also of the quadratic type and is negative definite. The construction of the Liapunov-Krasovskii functional is made according to the Kharitonov procedure described below.

Given positive definite $n \times n$ matrices P_0, P_j, R_j, $j = \overline{1, n}$ let us define on $\boldsymbol{C}([-\tau, 0], \boldsymbol{R}^n)$ the positive definite functional

$$W(\phi(\cdot)) = \phi^T(0) P_0 \phi(0) + \sum_{j=1}^{n} \phi^T(-\tau_j) P_j \phi(\tau_j) + \sum_{j=1}^{n} \int_{-\tau_j}^{0} \phi^T(\theta) R_j \phi(\theta) d\theta \qquad (17)$$

Since system (10) is exponentially stable, there exists a Liapunov-Krasovskii functional $V(\phi(\cdot))$ such that along the solutions of (10) we have the equality

$$\frac{d}{dt} V(z_t) = -W(z_t) \qquad (18)$$

Here we used the Krasovskii-Halanay-Hale notation $z_t(\cdot) = z(t+\cdot)$ for the state of the time-delay system.

The assumption about the tems k_i is much alike to the so-called *minimal stability* introduced by V. M.

Popov (Popov, V. M., 1966): in order to obtain stability for *all* nonlinear (and linear) functions from some sector, it is (minimally) necessary to have this property for *a single linear function* within this sector.

Since $A_0 = diag(-a_i)_1^n$ and $a_i > 0$, matrix A_0 is of the Hurwitz type. Therefore, taking also into account (8), we may take $k_i = 0$, $\underline{k_i} = 0$, $\overline{k_i} = L_i$.

The value of the "exact" Liapunov-Krasovskii functional for the trajectory segment $z\ (t + \cdot)$ is

$$V(z_t) = z^T(t)U(0)z(t) + \\ + \sum_{j=1}^{n} \int_{-\tau_j}^{0} z^T(t+\theta)\left[(\tau_j+\theta)R_j + P_j\right]z(t+\theta)d\theta \tag{19}$$

where, since the system (10) is exponentially stable, the matrix valued function

$$U(\tau) = \int_0^\infty K^T(t)\left[P_0 + \sum_{j=1}^{n}\left(P_j + \tau_j R_j\right)\right]K(t+\tau)dt \tag{20}$$

is well defined for all $\tau \in \boldsymbol{R}$ (see Kharitonov and Zhabko, 2001). In our case K(t) is the transition matrix of A_0 namely

$$K(t) = \exp(A_0 t) = diag(\exp(-a_i t)) \tag{21}$$

and $U(\tau)$ from (20) may be written as: $U(\tau) = U(0)\exp(A_0\tau)$

Following the steps in (Kharitonov and Zhabko, 2001), the time derivative of Liapunov-Krasovskii functional along the solutions of the perturbed system (13) is

$$\frac{d}{dt}V(y_t) = -W(y_t) + 2\left[\sum_{j=1}^{n}\Delta_j y(t+\tau_j)\right]^T U(0)y(t) \tag{22}$$

where W is given by (17). For the uncertainties we obtain the following quadratic restriction,

$$\Delta_j^T\Delta_j = \left(C_j diag(b_p)\right)^T C_j diag(b_p) \le \\ \le \left(\sum_{i=1}^{n} c_{ij}^2\right) b_j^2 I \le \left(\sum_{i=1}^{n} c_{ij}^2\right) L_j^2\, I = \rho_j I \tag{23}$$

which will be useful for the estimate of the perturbing term in (22) in order to still obtain a non-positive derivative of the Liapunov functional (19) along the perturbed system (13). It is not difficult, using standard inequalities (in the line of Gu, Kharitonov and Chen, 2003) to obtain the following estimate

$$\frac{d}{dt}V(y_t) \le -y^T(t)\left[P_0 - \mu U^T(0)U(0)\right]y(t) - \\ -\sum_{j=1}^{n} y^T(t-\tau_j)\left[P_j - \frac{2}{\mu}\rho_j I\right]y(t-\tau_j) - \\ -\sum_{j=1}^{n}\int_{-\tau_j}^{0} y^T(t+\theta)R_j y(t+\theta)d\theta \tag{24}$$

for some $\tilde{\mu} > 0$ and $\mu = \tilde{\mu}\sum_{j=1}^{n} L_j^2 \sum_{i=1}^{n} c_{ij}^2$. This is obtained from the definition of b_j and from the fact that $b_j \le L_j$ and with this the inequality (24) is fulfilled.

We have constructed a Liapunov - Krasovskii quadratic functional which is strictly positive definite and with the derivative along the linear system's solutions at least nonpositive; this last property is preserved with respect to the considered uncertainties and this shows a possible robust exponential stability of the linearized system (10). But, as already mentioned, the idea of Malkin (Barbashin, 1970; Malkin, 1952) gives more - exponential stability of the nonlinear system (7). In fact, if the Liapunov functional and its derivative - both being quadratic forms - have good sign properties for all $b_i \in \left(-\underline{k}_i, \overline{k}_i\right)$, then for any fixed $z_i \ne 0$ one can obtain b_i from (14) and for $b_i(z_i) \in \left(-\underline{k}_i, \overline{k}_i\right)$ the properties of the Liapunov functional do not change.

Remark that the terms b_i may be even time varying, what shows that f_i may be time varying within the range $\left(-\underline{k}_i, \overline{k}_i\right)$ provided they are at least integrable with respect to t. The integrability is necessary just to secure existence of the solution for the Cauchy problem. Also the Lipschitz property has now to hold uniformly with respect to t.

We have obtained the following result:

Theorem: *Let system (10) be exponentially stable. Then system (7) is exponentially stable for all nonlinearities of the form:*

$$f_i(z_i) = \left(k_i + b_i(z_i)\right)z_i \tag{25}$$

with $b_i(z_i)$ defined by (14), if there exists definite positive matrices P_0, P_j, R_j, $j = \overline{1,n}$ and a positive value μ, such that

$$P_0 > \mu U^T(0)U(0)\,,\ P_j > \frac{2}{\mu}\left(L_j^2\sum_{i=1}^{n} c_{ij}^2\right)I\,,\ R_j > 0 \tag{26}$$

Sketch of the proof: If we use the standard properties of the eigenvalues of positive definite matrices and the ordering of the quadratic forms, the following estimates are obtained

$$\delta\|\phi\|^2 \le V(\phi) \le \gamma\|\phi\|^2 \text{ , } \widetilde{W}(\phi) \ge \varepsilon W(\phi) \qquad (27)$$

for some positive δ, γ, ε and with $\|\phi\| = \sup\limits_{-\tau\le\theta\le 0} |\phi(\theta)|$, the usual norm of the uniform convergence on $\boldsymbol{C}(-\tau, 0; \boldsymbol{R}^n)$, where $\widetilde{W}$ is the right hand side of (24). From now on we have only to apply standard results of stability theory for time delay systems based on quadratic functionals (Halanay, 1963; Hale and Verduyn Lunel, 1993; Yoshizawa, 1966) to obtain the result.

Remark that we obtained *delay-independent stability criteria*. Also since both the Liapunov functional and its derivative are quadratic functionals the stability is *exponential* and since the functional and the inequality for the derivative are valid globally *the stability is also globally*.

4. CONCLUDING REMARKS AND OPEN PROBLEMS

The paper presents new results concerning stability of Hopfield-type neural networks with time delays and it is a continuation of our previous research concerning this type of recurrent neural networks (Danciu and Rasvan, 2000, 2001a, 2001b, 2005; Danciu, 2002, 2004). Since the approach have been based on the Liapunov method, we obtained only sufficient stability conditions. From this point of view our results are in the standard line. Their specific features come from the extended use of the methods of the absolute stability in the area of time delay systems with sector-restricted nonlinearities (Rasvan, 1975). It is this point of view that leads us to consider the approach of Malkin in the absolute stability applied to our case.

The genuine breakthrough from (Kharitonov and Zhabko, 2001; Gu, Kharitonov and Chen, 2003) allowed us to obtain improved sufficient stability criteria. We think this is due also to the context offered by this Liapunov functional construction to the approach of Malkin.

On the other hand, Hopfield-type neural networks are systems with several equilibria. This fact allows them to act as associative memories. An open problem in our future research is to find conditions for a desirable global behaviour of this type of recurrent neural networks affected by delays. We shall apply the same approach, but using a Liapunov functional whose derivative along the solutions of the nonlinear time-delayed perturbed system has to be canceled on the set of the equilibria and only there.

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