INTERVAL MATRIX SYSTEMS – INVARIANT SETS AND STABILITY PROPERTIES

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Abstract: A sufficient condition that is frequently used for exploring the exponential stability of an interval system requires the stability of a a unique test matrix, adequately built from the interval-type coefficients. We prove that the fulfilment of this sufficient condition guarantees a stronger property for the interval system, called diagonal stability, which, concomitantly with the standard exponential stability, ensures the flow (positive) invariance of certain time-dependent sets with respect to the state-space trajectories. Our approach covers both cases of discrete- and continuous-time interval systems.

Keywords: robustness, interval systems, diagonal stability, exponential stability, time-dependent invariant sets

1. INTRODUCTION

Consider the *interval matrix system* (IMS):

$$x'(t) = Ax(t);\ x(t_0) = x_0; \quad t, t_0 \in \mathbf{T},\ t \ge t_0,\ A \in A^I, \tag{1}$$

with discrete-time ($\mathbf{T} = \mathbf{Z}_+$) or continuous-time ($\mathbf{T} = \mathbf{R}_+$) dynamics and the operator ' acting accordingly, where A^I denotes the interval matrix:

$$A^I = \{A \in \mathbf{R}^{n\times n} :\ A^- \le A \le A^+\} \tag{2}$$

The two matrix inequalities in (2) have the componentwise meaning $a_{ij}^- \le a_{ij} \le a_{ij}^+,\ i, j = 1,\dots,n$, for $a_{ij}^-,\ a_{ij},\ a_{ij}^+$, representing the generic entries of the matrices $A^-,\ A,\ A^+$, respectively.

The following theorem provides *sufficient conditions* for the exponential stability of IMS (1), which are very easy to handle by direct computation, because their formulation relies on a unique test matrix $\overline{A}$ built from the bounds $a_{ij}^-,\ a_{ij}^+$ of A^I.

Theorem 1. (i) (Bauer and Premaratne 1993, Theorem 3), (Sezer and Šiljak, 1994, Corollary 1.2), (Kaszkurewicz and Bhaya, 2000, Theorem 3.4.17). In the discrete-time case, IMS (1) is exponentially stable, if the test matrix $\overline{A} = (\overline{a}_{ij})_{i,j=1,\dots,n}$ defined by:

$$\overline{a}_{ij} = \sup_{\mathbf{A}\in\mathbf{A}^I} \{|a_{ij}|\} = \max\{|a_{ij}^-|, |a_{ij}^+|\}, \quad i, j = 1,\dots,n, \tag{3i}$$

is Schur stable.
(ii) (Sezer and Šiljak, 1994, Corollary 2.2). In the continuous-time case, IMS (1) is exponentially stable if the test matrix $\overline{A} = (\overline{a}_{ij})_{i,j=1,\dots,n}$ defined by:

$$\begin{aligned}\overline{a}_{ii} &= \sup_{A\in A^I} \{a_{ii}\} = a_{ii}^+,\ i = 1,\dots,n,\\ \overline{a}_{ij} &= \sup_{A\in A^I} \{|a_{ij}|\} = \max\{|a_{ij}^-|, |a_{ij}^+|\}, \quad i \ne j,\ i, j = 1,\dots,n,\end{aligned} \tag{3ii}$$

is Hurwitz stable. ■

The current paper is going to prove that the Schur / Hurwitz stability of the test matrix $\overline{A}$ ensures to IMS (1) a stronger stability property than the standard exponential stability guaranteed by Theorem 1. This stronger property has been called by us *diagonal stability* and, concomitantly with the exponential stability of IMS (1), it guarantees the flow invariance (positive invariance) of certain time-dependent sets with respect to the trajectories of IMS (1). The current approach expands the researches initiated in (Pastravanu et al., 2005) for continuous-time systems. Moreover, we shall show that the diagonal stability of IMSs generalizes a concept used in linear algebra for referring to a special class of stable matrices, e.g. (Kaszkurewicz and Bhaya, 2000).

The presentation of our work continues according to the following plan: Section 2 gives the formal definition and some characterizations of the diagonally stable IMSs. Section 3 discloses the main results, namely the links between the test matrix $\overline{A}$ (3) and the diagonal stability of IMS (1), which are illustrated in Section 4. Some comments on the importance of these results are formulated in Section 5.

2. DIAGONALLY STABLE IMSs

Denote by $\| \ \|_p$ the Hölder *p*-norm in $\mathbf{R}^n$. If *D* is a positive diagonal matrix

$$D = \mathrm{diag}\{d_1,..., d_n\},\ d_i > 0,\ i = 1,...,n\ , \qquad (4)$$

let $\| \ \|_p^D$ stand for the vector norm given by $\|x\|_p^D = \left\|D^{-1}x\right\|_p$.

Definition 1. IMS (1) is *diagonally stable* in the Hölder *p*-norm (abbreviated as DS_p) if

(i) in the discrete-time case, there exist a positive diagonal matrix *D* and a constant $0 < r < 1$ such that

$$\forall \varepsilon > 0,\quad \forall t_0 \in \mathbf{Z}_+, \forall x_0 \in \mathbf{R}^n \text{ with } \|x_0\|_p^D \le \varepsilon$$
$$\Rightarrow \|x(t,t_0,x_0)\|_p^D \le \varepsilon r^{t-t_0}, \forall t \in \mathbf{Z}_+,\ t \ge t_0; \quad (5i)$$

(ii) in the continuous-time case, there exist a positive diagonal matrix *D* and a constant $r < 0$ such that

$$\forall \varepsilon > 0,\quad \forall t_0 \in \mathbf{R}_+, \forall x_0 \in \mathbf{R}^n \text{ with } \|x_0\|_p^D \le \varepsilon$$
$$\Rightarrow \|x(t,t_0,x_0)\|_p^D \le \varepsilon e^{r(t-t_0)}, \forall t \in \mathbf{R}_+,\ t \ge t_0. \ ■ (5ii)$$

Remark 1. Definition 1 is exactly the definition of the exponential stability of the equilibrium point {0}, e.g. (Michel and Wang, 1995, pp.107), formulated for the norm $\| \ \|_p^D$ and particularized for $\delta(\varepsilon) = \varepsilon$. If IMS (1) is DS_p, then it is also exponentially stable for any norm in $\mathbf{R}^n$, in the standard sense, with $\delta(\varepsilon) \le \varepsilon$. ■

The following theorem outlines the difference between the standard exponential stability and the diagonal stability.

Theorem 2. IMS (1) is DS_p *iff*

(i) in the discrete-time case, there exist a positive diagonal matrix *D* and a constant $0 < r < 1$ such that the time-dependent sets

$$S_c(t) = \left\{x(t) \in \mathbf{R}^n \ \middle|\ \|x(t)\|_p^D \le cr^t\right\},\ t \in \mathbf{Z}_+, c > 0\ , \quad (6i)$$

are flow invariant (positively invariant) with respect to the state space trajectories of IMS (1);

(ii) in the continuous-time case, there exist a positive diagonal matrix *D* and a constant $r < 0$ such that the time-dependent sets

$$S_c(t) = \left\{x(t) \in \mathbf{R}^n \ \middle|\ \|x(t)\|_p^D \le ce^{rt}\right\},\ t \in \mathbf{R}_+, c > 0 \quad (6ii)$$

are flow invariant (positively invariant) with respect to the state space trajectories of IMS (1).

Proof: Definition 1 is equivalent to the fact that each trajectory initialized at arbitrary $t_0 \in \mathbf{T}$ inside or on the boundary of the set $\left\{x \in \mathbf{R}^n \middle| \|x\|_p^D \le ch(t_0)\right\}$, $c > 0$, will remain inside or on the boundary of the set $\left\{x \in \mathbf{R}^n \middle| \|x\|_p^D \le ch(t)\right\}$, $c > 0$, for any $t \in \mathbf{T}$, $t \ge t_0$, where

$$h(t) = \begin{cases} r^t,\ t \in \mathbf{Z}_+ \ \text{(discrete-time case)}, \\ e^{rt}, t \in \mathbf{R}_+ \ \text{(continuous-time case)}. \end{cases} \quad ■ \ (7)$$

Remark 2. For the usual Hölder norms, namely $p = 1, 2, \infty$, the DS_p of IMS (1) shows that, for every $t \in \mathbf{T}$, the time-dependent invariant sets $S_c(t)$ have well-known shapes (hyper-diamonds for $p = 1$, hyper-ellipses for $p = 2$, and hyper-rectangles for $p = \infty$), whose symmetry axes coincide with the coordinate system, and their sizes are uniquely defined by the matrix *D* and the constants *c*, *r*. ■

The following two theorems provide important characterizations of the DS_p of IMS (1).

Theorem 3. IMS (1) is DS_p *iff* there exist a positive diagonal matrix *D* and a constant $0 < r < 1$ / $r < 0$ (discrete / continuous-time case) such that

$$\|\phi_A(\tau)\|_p^D \le h(\tau), \forall \tau \in \mathbf{T}, \forall A \in A^I, \qquad (8)$$

where $h(t)$ is defined by (7) and $\|\phi_A(\tau)\|_p^D$ denotes the operator norm induced by the vector norm $\| \ \|_p^D$, applied to the transition matrix

$$\phi_A(\tau) = \begin{cases} A^\tau, \text{(discrete-time case)}, \\ e^{A\tau}, \text{(continuous-time case)}. \end{cases} \qquad (9)$$

Proof. Sufficiency: If (8) is true, then, for $\forall\ t_0 \in \mathbf{T}$ and $\forall x_0, \|x_0\|_p^D \le h(t_0)$, we can write for $\forall A \in A^I$, $\|x(t,t_0,x_0)\|_p^D = \|\phi_A(t-t_0)x_0\|_p^D \le \|\phi_A(t-t_0)\|_p^D\ \|x_0\|_p^D \le h(t)$ $\forall t \in \mathbf{T}, t \ge t_0$.

Necessity: If IMS (1) is DS_p, for $\forall\ t_0 \in \mathbf{T}$, $\forall x_0$, $\|x_0\|_p^D = h(t_0)$, we can write $\|x(t,t_0x_0)\|_p^D \le h(t)$,

$\forall t \geq t_0$. On the other hand, for $\forall A \in A^I$, $\forall t_0, t \in \mathbf{T}$, $t \geq t_0$, we have $\| \phi_A(t-t_0) \|_p^D = \sup_{\|z_0\|_p^D = 1} \| \phi_A(t-t_0) z_0 \|_p^D =$

$$= \sup_{\|h^{-1}(t_0)x_0\|_p^D = 1} \left\| \phi_A(t-t_0) h^{-1}(t_0) x_0 \right\|_p^D =$$

$$= h^{-1}(t_0) \sup_{\|x_0\|_p^D = h(t_0)} \|x(t,t_0,x_0)\|_p^D \leq h(t-t_0) . \quad ■$$

Theorem 4. IMS (1) is DS_p *iff* there exist a positive diagonal matrix D and a constant $0 < r < 1$ / $r < 0$ (discrete / continuous-time case) such that $V(x) = \|x\|_p^D$ is a strong Lyapunov function with the decreasing rate r.

Proof: Obviously $V(x) = \| x \|_p^D$ is positive definite. The statement "V has the decreasing rate r" is equivalent to

$$\begin{aligned} &\forall t_0, t \in \mathbf{T},\ t_0 \leq t,\ \forall x_0 = x(t_0), \\ &V(x(t,t_0,x_0)) \leq h(t-t_0) V(x(t_0)), \end{aligned} \qquad (10)$$

where $h(t)$ is defined as in (7). Thus, we have to show that (10) is fulfilled *iff* IMS (1) is DS_p.

Necessity: If IMS (1) is DS_p, then $\forall A \in A^I$, $\forall t_0 \in \mathbf{T}$, $\forall x_0 = x(t_0) \in \mathbf{R}^n$, we have $\forall t \in \mathbf{T}$, $t \geq t_0$, $V(x(t,t_0,x_0)) = \| \phi_A(t-t_0) x(t_0) \|_p^D \leq \| \phi_A(t-t_0) \|_p^D \|x(t_0)\|_p^D \leq h(t-t_0) V(x(t_0))$, showing that condition (10) is met.

Sufficiency: We give a proof by contradiction and assume that IMS (1) is not DS_p. This means that $S_1(t)$ defined by (6) for $c = 1$ is not flow-invariant with respect to IMS (1), i.e. $\exists t_0, t \in \mathbf{T}$, $t_0 < t$, $h^{-1}(t_0)\|x(t_0)\|_p^D \leq 1 < h^{-1}(t)\|x(t)\|_p^D$. This contradicts (10) saying that $h^{-1}(t)\|x(t)\|_p^D \leq h^{-1}(t_0)\|x(t_0)\|_p^D$. ■

Based on the previous results, now we are able to express the link between the matrices $A \in A^I$ defining the dynamics of IMS (1) and the DS_p.

Theorem 5. IMS (1) is DS_p *iff* there exist a positive diagonal matrix D and a constant $0 < r < 1$ / $r < 0$ (discrete / continuous-time case) such that

(i) in the discrete-time case

$$\|A\|_p^D \leq r,\ \forall A \in A^I, \qquad (11\text{i})$$

(ii) in the continuous-time case

$$\mu_{\| \|_p^D}(A) \leq r,\ \forall A \in A^I, \qquad (11\text{ii})$$

where $\|A\|_p^D$ is the matrix norm induced by the vector norm $\| \|_p^D$ and $\mu_{\| \|_p^D}(A) = \lim_{\theta \downarrow 0} \left(\|I + \theta A\|_p^D - 1 \right) / \theta$ is a matrix measure based on the induced matrix norm, see, e.g. (Desoer and Vidyasagar, 1975).

Proof: (i) In the discrete time case, the *necessity* results from inequality (8) in Theorem 3 for $\tau = 1$. For *sufficiency*, note that inequality (11i) implies that for $\forall t \in \mathbf{Z}_+, \forall x \in \mathbf{R}^n$, if $x(t) = x$ then $\|x(t+1)\|_p^D = \|Ax(t)\|_p^D \leq \|A\|_p^D \|x(t)\|_p^D \leq r\|x(t)\|_p^D$, $\forall A \in A^I$, i.e. $V(x) = \|x\|_p^D$ is a strong Lyapunov function with the decreasing rate r.

(ii) In the continuous-time case, the *necessity* results from Theorem 2, because $\| \phi_A(\theta) \|_p^D \leq e^{r\theta}$ and $\mu_{\| \|_p^D}(A) = \lim_{\theta \downarrow 0} \left(\|I + \theta A\|_p^D - 1 \right) / \theta = \lim_{\theta \downarrow 0} \left(\|\phi_A(\theta)\|_p^D - 1 \right) / \theta \leq$
$\leq \lim_{\theta \downarrow 0} \left(e^{r\theta} - 1 \right) / \theta = r$.

Sufficiency: For $\forall t \in \mathbf{R}_+ \forall x \in \mathbf{R}^n$, if $x(t) = x$ then

$$D^+ \|x(t)\| = \lim_{\theta \downarrow 0} \left(\|x(t+\theta)\|_p^D - \|x(t)\|_p^D \right) / \theta =$$

$$= \lim_{\theta \downarrow 0} \left(\|\phi_A(\theta) x(t)\|_p^D - \|x(t)\|_p^D \right) / \theta \leq$$

$$\leq \left[\lim_{\theta \downarrow 0} \left(\|\phi_A(\theta)\|_p^D - 1 \right) / \theta \right] \|x(t)\|_p^D = \mu_{\| \|_p^D}(A) \|x(t)\|_p^D \leq$$

$\leq r\|x(t)\|_p^D$, $\forall A \in A^I$. Thus, $V(x) = \| x \|_p^D$ is a strong Lyapunov function with the decreasing rate r. ■

Remark 3. Theorem 5 confirms that DS_p is a stronger property than the standard exponential stability of IMS (1), which is characterized by the eigenvalues of the matrices $A \in A^I$. Denote by $\lambda_i(A)$ $i = 1, \ldots, n$, the n eigenvalues of $A \in A^I$. If IMS (1) is $\mathrm{DS}_{p\,r}$ then we have $|\lambda_i(A)| \leq \|A\|_p^D \leq r < 1$, $i = 1, \ldots, n$, $\forall A \in A^I$, (in the discrete-time case), and $\mathrm{Re}\, \lambda_i(A) \leq \mu_{\| \|_p^D}(A) \leq r < 0$, $i = 1, \ldots, n$, $\forall A \in A^I$, (in the continuous-time case). ■

Remark 4. For $p = 2$, inequalities (11i) and (11ii) in Theorem 4 yield the definition given in (Kaszkurewicz and Bhaya, 2000) for the diagonal stability of the matrices $A \in A^I$. Thus, in the discrete-time case, (11i) with $p = 2$ is equivalent with the Stein inequality fulfilled by the interval matrix A^I, i.e.

$$A^T D^2 A - r^2 D^2 \leq 0, \quad 0 < r < 1,\ \forall A \in A^I, \qquad (12\text{i})$$

and in the continuous-time case, (11ii) with $p = 2$ is equivalent with the Lyapunov inequality fulfilled by the interval matrix A^I, i.e.

$$A^T D^2 + D^2 A - 2rD^2 \leq 0, \quad r < 0,\ \forall A \in A^I. \qquad (12\text{ii})$$

This fact motivated us to use the concept of diagonal stability for IMS (1), by transferring it from the matrix algebra and enriching its sense within the context of the qualitative analysis of dynamical systems. ■

3. MAIN RESULTS

Theorem 6. IMS (1) is DS_p for any Hölder p-norm if

(i) in the discrete-time case, matrix $\overline{A}$ built according to (3i) is Schur stable;
(ii) in the continuous-time case, matrix $\overline{A}$ built according to (3ii) is Hurwitz stable.

To prove this result, we need the following two lemmas:

Lemma 1. (i) If P is a nonnegative matrix, then it has a real nonnegative eigenvalue (simple or multiple), denoted by $\lambda_{\max}(P)$, which dominates the whole spectrum of P, i.e. $|\lambda_i(P)| \le \lambda_{\max}(P)$, $i = 1,...,n$. (ii) If P is an essentially nonnegative matrix, then it has a real eigenvalue (simple or multiple), denoted by $\lambda_{\max}(P)$, which dominates the whole spectrum of P, i.e. $\mathrm{Re}(\lambda_i(P)) \le \lambda_{\max}(P)$, $i = 1,...,n$.
Proof: (i) $\lambda_{\max}(P)$ is the spectral radius of P. (ii) $\lambda_{\max}(sI+P)$ is the spectral radius of $sI+P$, where $s \ge p_{ii}$, $i = 1,\cdots,n$. ■

Lemma 2. (i) If P is a nonnegative matrix, then, for any $r > \lambda_{\max}(P)$, there exists a positive diagonal matrix $\Delta = \mathrm{diag}\{\delta^1,\ldots,\delta^n\}$ such that $\lambda_{\max}(P) \le \le \| \Delta^{-1}P\Delta \|_p < r$ for any Hölder p-norm. (ii) If P is an essentially nonnegative matrix, then, for any $r > \lambda_{\max}(P)$, there exists a positive diagonal matrix $\Delta = \mathrm{diag}\{\delta^1,\ldots,\delta^n\}$ such that $\lambda_{\max}(P) \le \le \mu_{\|\,\|_p}(\Delta^{-1}P\Delta) < r$ for any Hölder p-norm.
Proof: (i) If E is a square matrix with all its entries 1, then $\lambda_{\max}(P+\varepsilon E)$ as a function of $\varepsilon \ge 0$ is continuous and nondecreasing, according to Theorem 8.1.18 in (Horn and Johnson, 1985). Hence, for any $r > \lambda_{\max}(P)$, we can find an $\varepsilon^* > 0$ such that $\lambda_{\max}(P+\varepsilon^* E) \le r$. On the other hand, the matrix $P+\varepsilon^* E$ is positive and there exist its right and left Perron eigenvectors $v = [v_1 \ldots v_n]^T > 0$ and $w = [w_1 \cdots w_n]^T > 0$, respectively. If $1/p + 1/q = 1$, then, according to (Stoer and Witzgall, 1962), we can write $\| \Delta^{-1}(P+\varepsilon^* E)\Delta \|_p = \lambda_{\max}(P+\varepsilon^* E)$ with $\Delta = \mathrm{diag}\{v_1^{1/q}/w_1^{1/p},\cdots,v_n^{1/q}/w_n^{1/p}\}$, where the particular cases of norms $p=1$ and $p=\infty$ mean $1/p=1, 1/q=0$, and $1/p=0, 1/q=1$, respectively. Since any Hölder p-norm is monotonic, $\| \Delta^{-1}P\Delta \|_p < \| \Delta^{-1}(P+\varepsilon^* E)\Delta \|_p$ and, finally, we get $\lambda_{\max}(P) \le \| \Delta^{-1}P\Delta \|_p < r$. (ii) When P is essentially nonnegative, we consider $s > 0$ such that the matrix $sI+P$ is nonnegative and we conduct the proof similarly to (i). Note that when matrix P is irreducible, the positive diagonal matrix Δ can be built directly from the right and left Perron-Frobenius eigenvectors of P, yielding (i) $\lambda_{\max}(P) = \| \Delta^{-1}P\Delta \|_p$ and (ii) $\lambda_{\max}(P) = \mu_{\|\,\|_p}(\Delta^{-1}P\Delta)$, respectively. ■

Proof of Theorem 6: If $\overline{A}$ built according to (3i) is Schur stable, then Lemma 2 applied to $\overline{A}$ ensures the existence of a positive diagonal matrix D and of a constant $0 < r < 1$ such that $\| D^{-1}\overline{A}D \|_p < r < 1$, for any Hölder p-norm. On the other hand, for any $A \in A^I$ we have the componentwise matrix inequality $D^{-1}AD \le | D^{-1}AD | = D^{-1} | A | D \le D^{-1}\overline{A}D$. This yields $\| D^{-1}AD \|_p \le \| D^{-1}\overline{A}D \|_p$, since any Hölder p-norm is monotonic. Thus, we get $\| A \|_p^D = \| D^{-1}AD \|_p < r < 1$, $\forall A \in A^I$, showing that IMS (1) is DS_p. (ii) If $\overline{A}$ built according to (3ii) is Hurwitz stable, then we use a similar construction to prove the existence of a positive diagonal matrix D and of a constant $r < 0$ such that $\forall A \in A^I$, $\mu_{\|\,\|_p^D}(A) = \mu_{\|\,\|_p}(D^{-1}AD) \le \mu_{\|\,\|_p}(D^{-1}\overline{A}D) < r < 0$, i.e. IMS (1) is DS_p. ■

Remark 5: The usage of Lemma 2 in the proof of Theorem 6 provides a procedure for finding the positive diagonal matrix D and the constant $0 < r < 1$ (discrete-time) / $r < 0$ (continuous-time) which define the time-dependent sets that are flow-invariant with respect to the trajectories of IMS (1). ■

Theorem 7. The Schur / Hurwitz stability of matrix $\overline{A}$ built according to (3i) / (3ii) is also a necessary condition for IMS (1) to be DS_p if there exists $A^* \in A^I$ such that
(i) in the discrete-time case $\| A^* \|_p^D = \| \overline{A} \|_p^D$;
(ii) in the continuous-time case $\mu_{\|\,\|_p^D}(A^*) = \mu_{\|\,\|_p^D}(\overline{A})$.
Proof: If IMS (1) is DS_p with the positive diagonal matrix D and the constant $0 < r < 1$ (discrete-time) / $r < 0$ (continuous-time), then (i) in the discrete-time case, condition (11i) implies $\lambda_{\max}(\overline{A}) \le \| \overline{A} \|_p^D = \| A^* \|_p^D < 1$; (ii) in the continuous-time case, condition (11ii) implies $\lambda_{\max}(\overline{A}) \le \mu_{\|\,\|_p^D}(\overline{A}) = \mu_{\|\,\|_p^D}(A^*) < 0$. ■

A direct consequence of Theorem 7 is the following:

Corollary 1. The Schur/Hurwitz stability of matrix $\overline{A}$ built according to (3i)/(3ii) is a necessary and sufficient condition for IMS (1) to be DS_1 and DS_∞. ■

Remark 6. The diagonal stability of IMS (1) in norm $p=\infty$ (DS_∞) is the *componentwise exponential asymptotic stability* (CWEAS) of IMS (1), which was analyzed by the previous works (Pastravanu and Voicu, 1999), (Pastravanu and Voicu, 2002), (Pastravanu and Voicu, 2004). Unlike the current approach, the aforementioned papers exploit the background created by (Voicu, 1987), (Voicu, 1984) and characterize CWEAS of IMS (1) by $\bar{A}d \le rd$, $d=[d_1 \cdots d_n]^T > 0$ and $0<r<1$ (discrete-time) / $r<0$ (continuous-time). It is a straightforward task to prove the equivalence between these conditions and $\| \bar{A} \|_p^D \le r < 1$ (discrete-time) / $\mu_{\| \, \|_p^D}(\bar{A}) \le r < 0$ (continuous-time) for $p=\infty$, $D=\mathrm{diag}\{d_1,\dots,d_n\}$.■

4. ILLUSTRATIVE EXAMPLE

Example 1 (Xin, 1987), (Chen, 1992): Consider the continuous-time IMS defined by the interval matrix:

$$A^I = \{A \in \mathbf{R}^{2\times 2}: \;\; A^- \le A \le A^+\},$$
$$A^- = \begin{bmatrix} -5 & 1 \\ 4 & -6 \end{bmatrix}, \; A^+ = \begin{bmatrix} -3 & 2 \\ 5 & -4 \end{bmatrix}, \qquad (13)$$

for which the test matrix $\bar{A}$ built in accordance with (3ii):

$$\bar{A} = \begin{bmatrix} -3 & 2 \\ 5 & -4 \end{bmatrix} \qquad (14)$$

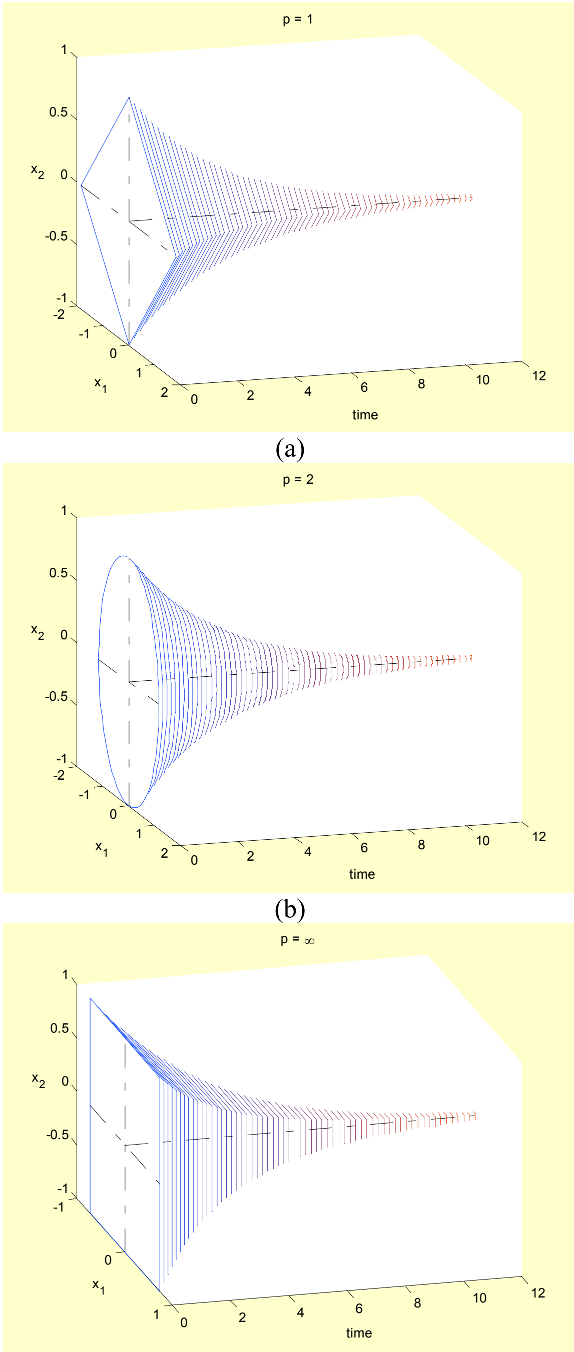
is Hurwitz stable since $\lambda_{\max}(\bar{A}) = (-7+\sqrt{41})/2 < 0$. Unlike papers (Xin, 1987), (Chen, 1992), proving that the IMS is exponentially stable, our Theorem 6 guarantees that the IMS has a stronger property, namely it is DS_p for any Hölder *p*-norm.

Given a Hölder *p*-norm and considering $r=\lambda_{\max}(\bar{A})$, there exist a positive diagonal matrix D_p such that the time-dependent set

$$S^p(t) = \left\{ x(t) \in \mathbf{R}^2 \,\middle|\, \left\| D_p^{-1} x(t) \right\|_p \le e^{rt} \right\}, \; t \in \mathbf{R}_+, \qquad (15)$$

defined according to (6ii), is flow invariant with respect to all the trajectories of the IMS. Note that matrix $\bar{A}$ (14) is irreducible, its right and left Perron-Frobenius eigenvectors being $v = \left[(1+\sqrt{41})/10 \;\; 1 \right]^T$ and $w = \left[(1+\sqrt{41})/4 \;\; 1 \right]^T$, respectively. Taking into account the proof of Lemma 2, matrix D_p can be constructed with the components of v and w as follows: for $p=1$, $D_1 = \mathrm{diag}\{w_1, w_2\}$, for $p=2$, $D_2 = \mathrm{diag}\{\sqrt{v_1 w_1}, \sqrt{v_2 w_2}\}$ and for $p=\infty$, $D_\infty = \mathrm{diag}\{v_1, v_2\}$. Figure 1 shows the graphical representations of the three invariant sets.

Example 2: The same test matrix $\bar{A}$ as in Example 1 results from the construction procedure (3ii) for an IMS with a larger matrix interval, namely:



(a) (b) (c)

Fig. 1. 3D visualisation of the invariant sets defined by (15) for: (a) $p=1$; (b) $p=2$; (c) $p=\infty$.

$$A^I = \{A \in \mathbf{R}^{2\times 2}: \;\; A^- \le A \le A^+\},$$
$$A^- = \begin{bmatrix} -10^{10} & -2 \\ -5 & -10^{10} \end{bmatrix}, \; A^+ = \begin{bmatrix} -3 & 2 \\ 5 & -4 \end{bmatrix}, \qquad (16)$$

which includes the matrix interval defined by (13). All the comments on the IMS dynamics formulated in Example 1 preserve their validity, despite the broader range covered by the current A^I.

Example 3: Enlarge the interval matrix in Example 1, by maintaining the same A^+ and taking

$$A^- = \begin{bmatrix} -5 & -3 \\ -3 & -6 \end{bmatrix}. \qquad (17)$$

It is a straightforward task to check that the current IMS is exponentially stable, since the roots of the characteristic polynomial

$$\det(sI - A) = s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21}$$

are located in the left half-plane of the complex plane for any matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$
$$a_{11} \in [-5,-3],\ a_{12} \in [-3,2],$$
$$a_{21} \in [-3,5],\ a_{22} \in [-6,-4].$$

However, Corollary 1 shows that the IMS is neither DS_1 nor DS_∞, because the test matrix $\overline{A}$ is Hurwitz unstable.

5. CONCLUSIONS

Our work proves that the Schur / Hurwitz stability of the test matrix $\overline{A}$ built according to (3i) / (3ii) ensures the DS_p of IMS (1) for any Hölder p-norm. This is a stronger property than the standard exponential stability of IMS (1), because it also guarantees the flow-invariance of certain time-dependent sets. The usage of Lemma 2 in the proof of Theorem 6 provides a procedure for finding these time-dependent flow-invariant sets, whenever the test matrix $\overline{A}$ is Schur / Hurwitz stable. Under the hypothesis of Theorem 7, the Schur / Hurwitz stability of the test matrix $\overline{A}$ becomes a necessary and sufficient condition for IMS (1) to be DS_p. Thus we can rediscover the CWEAS property of IMS (1) studied by some previous works of ours as a particular case of DS_p given by $p = \infty$.

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