# Absolute stability conditions for some scalar nonlinear time-delay systems with monotone increasing nonlinearity ⋆

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**Abstract:** This paper deals with the analysis of the absolute stability for a class of scalar nonlinear time-delay systems having monotone increasing nonlinearities. The approach is based on frequency domain inequalities of Popov type for time-delay systems in the critical case of the transfer function with a simple zero pole. In order to solve the *minmax* problem which arises from the frequency domain inequality, the analytical analysis was completed by numerical computations using MATLAB software package. We have obtained a *time-delay dependent absolute stability condition*.

*Keywords:* Absolute stability, Frequency domain inequality, Limit stability, Time-delay system

## 1. STATE OF THE ART AND PROBLEM STATEMENT

### 1.1 Absolute stability and frequency domain inequalities

The property of absolute stability had been motivated by Letov (1955); Letov and Lurie (1957) by the rather poor information about nonlinearity and, as Răsvan (2002) remarks, "in a more contemporary statement this is nothing else but robust stability with respect to some kind of nonlinear function uncertainty".

Absolute stability refers to the global asymptotic stability of the zero equilibrium of the nonlinear system

$$\dot{x}(t) = Ax - b\varphi(c^*x) \tag{1}$$

having sector restricted nonlinearities of the form (see Fig. 2)

$$0 \le \underline{\varphi} \le \frac{\varphi(\sigma)}{\sigma} \le \overline{\varphi} \le +\infty\ ,\ \varphi(0) = 0 \tag{2}$$

the property of the equilibrium being valid for all the linear and nonlinear functions verifying (1).

For the absolute stability problem, the system under the analysis (1) might be written as a "negative" feedback connection (Fig. 1) of the linear block L described by

$$\begin{aligned} \dot{x}(t) &= Ax + b\mu_1 \\ \sigma_1 &= c^*x \end{aligned} \tag{3}$$

with the nonlinear block, subject to (2) and described by

$$\sigma_2 = \varphi(\mu_2) \tag{4}$$

the interconnection rules being

$$\mu_2 = \sigma_1\ ,\ \mu_1 = -\sigma_2. \tag{5}$$

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The above remark will be useful in the next sections. In the sequel we shall use the following notations regarding the signals in Fig. 1: $\mu := \mu_1 = -\sigma_2$ and $\sigma := \sigma_1 = \mu_2$.

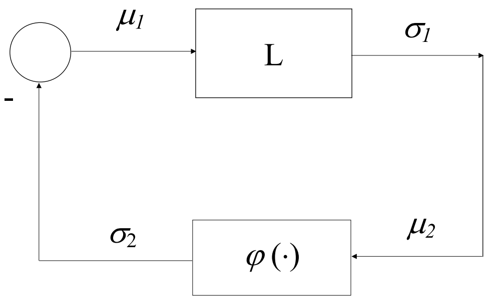


Fig. 1. Absolute stability feedback structure.

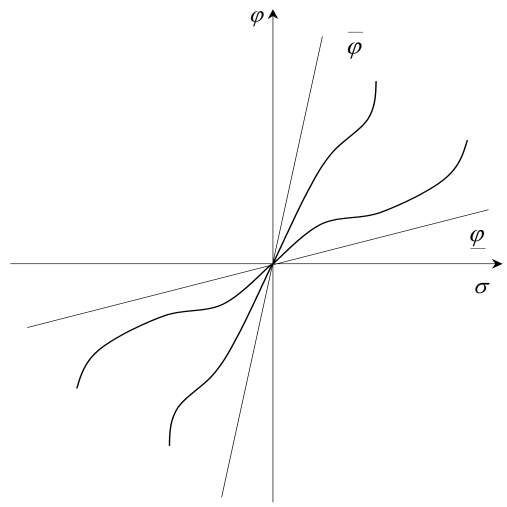


Fig. 2. Sector restricted nonlinearities

Concerning the system with the structure in Fig. 1 we recall here the so-called Aizerman problem (Răsvan et al. (2010)). Let L be a linear controlled system. If $\varphi(\sigma)$ is a linear function $\varphi(\sigma) = h\sigma$, any stability criterion would give a sector

$h \in (\underline{\varphi}_H, \overline{\varphi}_H)$ called the Hurwitz sector which, corresponding to the necessary and sufficiently stability conditions, is thus maximal. If, on the other hand, one considers nonlinear functions verifying (2), then only sufficient global stability conditions can be obtained (generally speaking) and the resulting maximal sector will be, as a rule, more narrow than the Hurwitz sector. *The comparison of the two sectors is the Aizerman problem*: the closer they are, the less is the "degree of conservatism" obtained via the available sufficient conditions of absolute stability (among which the frequency domain inequalities together with the Liapunov functions and the Linear Matrix Inequalities associated to them are the less conservative).

According to Lefschetz (1965), the development of the absolute stability theory has known two periods: the pre-Popov period and Popov period that started after 1960. Regarding the second period, this is marked by the frequency domain inequalities used for analyzing the stability of nonlinear systems. These were introduced by the Romanian scientist V. M. Popov in his pioneering paper (Popov (1959)) and became known worldwide especially after his seminal paper (Popov (1961)).

For the goal of the paper we shall introduce here the absolute stability result based on frequency domain inequality of Popov-type for time-delay systems in the critical case of a zero root (a straightforward extension of the Theorem 6.1 in Răsvan (1975)).

**Theorem 1.** Consider the control system described by

$$\begin{gathered}\dot{x}(t) = Ax(t) + \sum_1^r B_k x(t-\tau_k) + b_0\xi(t) + \sum_1^r b_k \xi(t-\tau_k),\\ \dot{\xi}(t) = -\varphi(\sigma(t)),\\ \sigma(t) = c_0^{'} x(t) + \sum_1^r c_k^{'} x(t-\tau_k) + \gamma_0 \xi(t) + \sum_1^r \gamma_k \xi(t-\tau_k)\end{gathered} \tag{6}$$

where $\varphi(\sigma)$ is a continuous nonlinearity verifying the conditions:

$$0 < \frac{\varphi(\sigma)}{\sigma} < k \leq +\infty\ ,\ \varphi(0) = 0. \tag{7}$$

One supposes that the characteristic equation

$$\det\left(sI - A - \sum_1^r B_k e^{-s\tau_k}\right) = 0 \tag{8}$$

has all its roots within $\mathbb{C}^-$, and

$$\gamma_0 + \sum_1^r \gamma_i - \left(c_0^{'} + \sum_1^r c_j^{'}\right)\left(A + \sum_1^r B_k\right)^{-1}\left(b_0 + \sum_1^r b_l\right) > 0. \tag{9}$$

If there exists $q \geq 0$ finite such that

$$\frac{1}{k} + \Re e\ (1 + \imath\omega q) H(\imath\omega) \geq 0, \tag{10}$$

where

$$\begin{aligned}H(s) = \frac{1}{s}\Bigg[\gamma_0 + \sum_1^r \gamma_i e^{-s\tau_i} + \left(c_0^{'} + \sum_1^r c_j^{'} e^{-s\tau_j}\right)\cdot\\ \left(sI - A - \sum_1^r B_k e^{-s\tau_k}\right)^{-1}\left(b_0 + \sum_1^r b_l e^{-s\tau_l}\right)\Bigg],\end{aligned} \tag{11}$$

then the system (6) is asymptotic stable for every nonlinearity within the class above defined.

*Remark 2.* The Theorem 1 ensures absolute stability for all nonlinearities which verify (7), yielding the absolute stability sector $(0,k)$ which can be compared with the Hurwitz sector in the Aizerman problem; the fulfilment of the frequency domain inequality (10) will give the "dimension" of the absolute stability sector. The inequality (9) ensures the limit stability property of the linear block in the critical case considered and, is similarly to the condition (12) in Theorem 4 from the subsection *Limit stability property* which will follow. The equation (11) is nothing else that the transfer function of the system (6).

### *1.2 The limit stability property*

In connection with absolute stability and Aizerman problem is the limit stability property - introduced by Aizerman and Gantmakher (1963). Consider again the systems having the structure in Fig. 1 and suppose that the nonlinear function is known with some uncertainty. As already said, we call absolute stability a robust version of the stability property, i.e. the stability of the zero equilibrium for all nonlinear and linear function within the sector $(0,k)$ and belonging to a certain class of functions. For ensuring the absolute stability property, a necessary and minimal condition would be exponential stability for a single linear function of the sector. In particular, if L defines an exponentially stable linear system, the property (called by Popov (1973) minimal stability) holds for $\varphi(\sigma) \equiv 0$. But if L is in a critical case i.e. it has the spectrum in $\mathbb{C}^-$ as well as on $\imath\mathbb{R}$ a necessary (and minimal) requirement would be exponential stability for $\varphi(\sigma) = \varepsilon\sigma$ with $0 < \varepsilon < \varepsilon_0$ and $\varepsilon_0 > 0$ arbitrarily small. This is called limit stability and is in fact a property of the linear block L; the rigorous definition would be as follows

*Definition 3.* Let L be a linear dynamical block described by some input/output operator - a convolution (in the time domain) or a transfer function (in the complex domain) - connecting the input $\mu$ to the output $\sigma$. System L is said to have the limit stability property if it is output stabilizable by the output feedback $\mu = -\varepsilon\sigma$ with $0 < \varepsilon < \varepsilon_0$ and $\varepsilon_0 > 0$ arbitrarily small.

There are known necessary and sufficient conditions for limit stability for linear block L associated to a strictly proper rational transfer function - see Aizerman and Gantmakher (1963). They were extended to "the delay case", i.e. to linear blocks whose transfer functions are strictly proper meromorphic functions defined by a ratio of quasi-polynomials; for the sake of the completeness, we shall give in the sequel the main result for this case (Răsvan (1975)).

**Theorem 4.** Consider a linear block with the transfer function $H(s) = N(s)/D(s)$ where $N(s)$ and $D(s)$ are quasi-polynomials, $H(s)$ being strictly proper in the sense that the degree of the principal term of $D(s)$ is larger than that of $N(s)$ and assume $D(s)$ to have at most a finite number of roots on $\imath\mathbb{R}$, the other roots being in $\mathbb{C}^-$. For the limit stability of this system it is necessary and sufficient that the multiplicity of each pole should be at most 2 and the following conditions hold for it

a) for a simple zero pole

$$\Re es\, H(s)\,|_{s=0} > 0 \tag{12}$$

b) for a simple non-zero pole, if the Laurent expansion is considered

$$\begin{aligned} \Re e\, H(\imath\omega) &= \frac{e_{-1}}{\omega-\omega_0} + d_0 - e_1(\omega-\omega_0) - \\ &-d_2(\omega-\omega_0)^2 + \ldots \\ \Im m\, H(\imath\omega) &= \frac{-d_{-1}}{\omega-\omega_0} + e_0 + d_1(\omega-\omega_0) - \\ &-e_2(\omega-\omega_0)^2 + \ldots \end{aligned} \tag{13}$$

and the sequence: $d_{-1}$; $e_{-1}e_0$; $-d_1$; $-e_1e_2$; $d_3$; ... is associated, either $d_{-1} \neq 0$ or $e_{-1} \neq 0$ and the first non-zero term of the above sequence should be strictly positive;

c) for a double zero root, if the Laurent expansion is considered

$$\begin{aligned} \Re e\, H(\imath\omega) &= \frac{-d_{-2}}{\omega^2} + d_0 - d_2\omega^2 + \ldots \\ \Im m\, H(\imath\omega) &= \frac{-d_{-1}}{\omega} + d_1\omega - d_3\omega^2 + \ldots \end{aligned} \tag{14}$$

and the sequence $-d_{-1}$; $d_1$; $-d_3$; ... is associated, then *i*) $d_{-2} > 0$ and *ii*) the first non-zero term of the sequence should be strictly negative;

d) for a double non-zero root, if the Laurent expansion is considered

$$\begin{aligned} \Re e\, H(\imath\omega) &= \frac{-d_{-2}}{(\omega-\omega_0)^2} - \frac{e_{-1}}{\omega-\omega_0} + d_0 - \\ &-e_1(\omega-\omega_0) - \ldots \\ \Im m\, H(\imath\omega) &= \frac{-e_{-2}}{(\omega-\omega_0)^2} - \frac{d_{-1}}{\omega-\omega_0} + e_0 + \\ &+d_1(\omega-\omega_0) - \ldots \end{aligned} \tag{15}$$

and the sequence $d_{-1}$; $-e_0^2$; $-d_1$; $-e_2^2$; ... is associated, then *i*) $d_{-2} > 0$, $e_{-2} = 0$ and *ii*) the first non-zero element of the sequence should be strictly positive.

Within the theorem the notion "principal term" is used in the sense of Pontryagin: if

$$h(z,w) = \sum_{m,n} a_{mn} z^m w^n$$

is a polynomial in two variables, $a_{rs}z^r w^s$ is the principal term of the polynomial if $a_{rs} \neq 0$ and for any other term with $a_{mn} \neq 0$ one of the following is possible: i) $r > m$, $s > n$; ii) $r = m$, $s > n$; iii) $r > m$, $s = n$. Note that any quasi-polynomial may be given the form $h(z, e^z)$ with $h(z,w)$ a polynomial in two variables.

### *1.3 The mathematical model and the problem statement*

Consider the class of scalar nonlinear time-delay systems described by

$$\dot{x}(t) = a - bx(t-\tau)\psi(x(t-\tau)) \;,\; a > 0 \;,\; b > 0 \tag{16}$$

where $\psi(\cdot)$ is a monotone increasing nonlinearity, $\tau > 0$ is a constant time-delay and the initial condition $x(\theta) = \varphi(\theta)$, for $\theta \in [-\tau, 0]$, where $\varphi \in \mathscr{C}(-\tau, 0; \mathbb{R})$.

The differential equation (16) may model, for instance, the fluid dynamics in high-performance networks. We refer here to the mathematical model proposed in Kelly (2001) for describing the rate control algorithm in order to avoid the congestion in communication networks. Considering the case when a collection of flows uses a single resource and shares the same gain parameter $K$, the model of Kelly (2001) reads as

$$\dot{y}(t) = K[w - y(t-\tau)p(y(t-\tau))] \;,\; K > 0 \;,\; w > 0 \tag{17}$$

where $\tau > 0$, assumed constant, represents the round-trip time and $p(\cdot)$ can be viewed as "the probability a packet produces a congestion indication signal", thus being "positive, continuous, strictly increasing function of $y$ and bounded above by unity" (Kelly (2001)).

We shall turn now to the mathematical model (16). Let $\bar{x}$ be the unique equilibrium of (16), thus verifying $a = b\psi(\bar{x})\bar{x}$. Using a change of the coordinates, $\xi = x - \bar{x}$, one can shift the equilibrium point $\bar{x}$ to the origin so that the system (16) can be written into the form:

$$\dot{\xi} = -b\phi(\xi(t-\tau)) \tag{18}$$

where the nonlinear function

$$\phi(\xi) = \psi(\bar{x}+\xi)(\bar{x}+\xi) - \psi(\bar{x})\bar{x}. \tag{19}$$

verifies the conditions

$$\phi(0) = 0 \;,\; \frac{\phi(\xi)}{\xi} > 0. \tag{20}$$

Instead (18), we shall consider - without loss of the generality - the nonlinear system

$$\dot{\xi} = -\phi(\xi(t-\tau)) \tag{21}$$

with $\tau > 0$ and $\xi(\theta) = \chi(\theta)$, for $\theta \in [-\tau, 0]$, where $\chi \in \mathscr{C}(-\tau, 0; \mathbb{R})$.

The aim of this paper is to obtain conditions for absolute stability of the class of systems described by (21), where the nonlinearity $\phi(\cdot)$ is a monotone increasing function verifying (20).

## 2. THE ABSOLUTE STABILITY RESULT

### *2.1 The frequency domain condition*

Following the absolute stability approach which is described in section I, the system (21) can be written as a negative feedback connection of the linear block

$$\begin{aligned} \dot{\xi}(t) &= \mu(t) \\ \sigma(t) &= \xi(t-\tau) \end{aligned} \tag{22}$$

with a nonlinear one, the interconnection rule being

$$\mu(t) = -\phi(\sigma(t)). \tag{23}$$

On the other hand, the system (21) is of the form (6) with $A = b_0 = c_0 = \gamma_0 = 0$ and $B_k = b_k = c_k = 0$ for $k = \overline{1,r}$, $\gamma_1 := \gamma = 1$. In both cases the transfer function of the linear block results

$$H(s) = \frac{e^{-\tau s}}{s}, \tag{24}$$

and one remarks the system is in the critical case a) of the Theorem 4: a simple zero pole. It can be easily seen that the linear block has the limit stability property since

$$\Re es\, H(s)|_{s=0} = 1 > 0 \tag{25}$$

or, equivalent, it is verified the condition (9).

The frequency domain inequality (10) is written in our case as

$$\frac{1}{k} + \Re e\,(1 + \imath\omega\theta)\frac{e^{-\imath\omega\tau}}{\imath\omega} = \frac{1}{k} + \left(\theta\cos\omega\tau - \frac{\sin\omega\tau}{\omega}\right) > 0 \tag{26}$$

and, dividing by $\tau > 0$ one obtains

$$\frac{1}{k_0} + \theta_0\cos\lambda - \frac{\sin\lambda}{\lambda} > 0\ ,\ \forall\lambda > 0 \tag{27}$$

where we denoted $\lambda := \omega\tau$, $\theta_0 := \frac{\theta}{\tau}$ and $k_0 := k\tau$. The inequality (27) can be written as a *minimax* problem

$$\max_{\theta_0 \geq 0}\min_{\lambda \geq 0}\left(\theta_0\cos\lambda - \frac{\sin\lambda}{\lambda}\right) > -\frac{1}{k_0}. \tag{28}$$

Let

$$f(\lambda) = \theta_0\cos\lambda - \frac{\sin\lambda}{\lambda}\ ,\ \forall\lambda > 0 \tag{29}$$

be the function under evaluation. On can see that $f(0) = \theta_0 - 1$ and the condition $f(0) > -\frac{1}{k_0}$ will give $\theta_0 > 1 - \frac{1}{k_0}$. For $\lambda_d \to \infty$ and $\lambda_d = n\pi$ the most unfavorable case is $\cos\lambda_d = -1$ and $f(\lambda_d) = -\theta_0 > -\frac{1}{k_0}$ gives $\theta_0 < \frac{1}{k_0}$. We have obtained the general boundary conditions:

$$1 - \frac{1}{k_0} < \theta_0 < \frac{1}{k_0} \tag{30}$$

and one can see that the alternate sign of the function impose as necessary the condition $k_0 < \infty$. Also, one remarks the inequality (30) shows that the larger $\theta_0$ is the smaller is $k_0$ and as a consequence the narrow is the absolute stability sector.

A summary analysis shows that the interval $(0,\pi)$ is interesting for the variation of function $f$. First of all, one observes that the second term in (29) is the function $\mathrm{sinc}(\lambda) = \sin\lambda/\lambda$ whose principal lobe has positive values on $(0,\pi/2)$ and due to the minus sign it has a unfavorable influence for our *minimax* problem. On the other hand, for $\lambda \in (\pi/2,\pi)$ both terms of the function have negative values. Evaluating $f(\pi/2) = -2/\pi > -1/k_0, \forall\theta_0$ we obtain an estimation for $k_0$:

$$1 - \frac{1}{k_0} < 1 - \frac{2}{\pi} < \frac{2}{\pi} < -\frac{1}{k_0}. \tag{31}$$

### 2.2 *The analysis of the function on intervals*

Concerning the *minimax* problem, the general analysis of the function and its derivative

$$f'(\lambda) = -\theta_0\sin\lambda - \frac{\lambda\cos\lambda - \sin\lambda}{\lambda^2} \tag{32}$$

leads to the following remarks:

- $\lambda = n\pi$: $f(n\pi) = (-1)^n\theta_0$ which shows again that $\theta_0$ cannot be too large since for $n = (2p+1), p = 0,1,\ldots$ it would give a more restrictive condition for the absolute stability sector: $k_0 < \frac{1}{\theta_0}$.
- $\lambda = n\frac{\pi}{2}$: $f((2p+1)\pi + \frac{\pi}{2}) > 0$ and it is not important for the problem; $f(2p\pi + \frac{\pi}{2}) = -\frac{1}{2p\pi+\frac{\pi}{2}} < 0$ and the smallest value (having the maximum modulus) is $f(\frac{\pi}{2}) = -\frac{2}{\pi}$.

A. Consider now the interval $\lambda \in (0,\pi)$.

a) $\lim_{\lambda\to 0_+} f'(\lambda) = 0$ which means $\lambda = 0$ is an extremum. Making use of Taylor expansion we obtain $f'(\lambda) = (\frac{1}{3} - \theta_0)\lambda + o(\lambda^2)$ and we deduce that on $(0,\pi)$: i) if $0 \leq \theta_0 < \frac{1}{3}$ then $\lambda = 0$ is a minimum and ii) if $\theta_0 > \frac{1}{3}$ then $\lambda = 0$ is a maximum.

b) Evaluating the derivative in $\lambda = \frac{\pi}{2}$ we obtain an other point of interest for our analysis on $\theta_0$: $\frac{4}{\pi^2}$. For $\theta_0 = \frac{4}{\pi^2}$, $f(\frac{\pi}{2}) = -\frac{2}{\pi}$ is a minimum on $(0,\pi)$ and $f$ is strictly increasing for $\theta_0 < \frac{4}{\pi^2}$ and decreasing otherwise.

We thus have obtained the following intervals of interest for $\theta_0$ when $\lambda \in (0,\pi)$:

- $\theta_0 \in (0,\frac{1}{3})$: $f(0) = \theta_0 - 1 < \frac{2}{3}$ is a minimum on $(0,\pi)$.
- $\theta_0 \in (\frac{1}{3},\frac{4}{\pi^2})$: $f(0)$ is a maximum on $(0,\pi)$ and there exists a minimum in $(0,\frac{\pi}{2})$, then the function rises; $f(\pi) = -\theta_0 < 0$; $f'(\pi) = \frac{1}{\pi} > 0, \forall\theta_0$.
- $\theta_0 \in (\frac{4}{\pi^2},\infty)$: the minimum on $(0,\pi)$ is within the interval $(\frac{\pi}{2},\pi)$ and $f(\overline{\lambda}_0) < -\frac{2}{\pi}$.

B. The analysis on the intervals $(2p\pi,(2p+1)\pi)$, $p = 1,2,\ldots$ gives the following conclusions:

- $\theta_0 < \frac{1}{(2p\pi+\frac{\pi}{2})^2} < \frac{1}{3}$: $f$ has a negative minimum within the interval $(2p\pi, 2p\pi + \frac{\pi}{2})$.
- $\theta_0 > \frac{1}{(2p\pi+\frac{\pi}{2})^2}$: $f$ has a negative minimum within the interval $(2p\pi + \frac{\pi}{2},(2p+1)\pi)$.

C. On the intervals $((2p+1)\pi,(2p+2)\pi)$, $p = 1,2,\ldots$ the extremum points are positive maxima and, are not of interest for our analysis.

We conclude that the intervals of interest for checking the minima have the general form

$$\lambda \in (2p\pi,(2p+1)\pi),\ \ p = 0,1,\ldots. \tag{33}$$

Let $\overline{\lambda}_p$ be the solution of $f'(\lambda) = 0$ on such an interval; it verifies also

$$\tan \lambda_p = \frac{\lambda_p}{1-\theta_0 \lambda_p^2} \tag{34}$$

and one can compute the minimum on the interval

$$f(\lambda_p) = \left[\theta_0(1-\theta_0\lambda_p^2)\right] \frac{\sin\lambda}{\lambda} < 0 \tag{35}$$

Since on such intervals $\text{sinc}(\lambda) > 0$, it results that for $p \geq 1$ this minimum is negative no matter where it is placed: either within the first or second half of the interval.

We can estimate now that the larger $\theta_0 > 0$ is the smaller the negative minimum will be. This means that $\theta_0$ cannot be increased too much since the *minimax* problem requires the maximization of the minima with respect to $\theta_0$.

*2.3 The analysis of the local minima on intervals*

The sequence of the local minima is defined by the equation

$$(1-\theta_0\lambda^2)\sin\lambda - \lambda\cos\lambda = 0 \tag{36}$$

with the solution defined by (34) and the value of a local minimum

$$f(\lambda_p) = \left(\theta_0 + \frac{1}{\theta_0\lambda_p^2 - 1}\right)\cos\lambda_p. \tag{37}$$

The analysis on each interval of interest for $\theta_0$, namely those we have found in section *2.2* (A and B), is simple but tediously. It reveals that the function of the minima is the same in all cases and has the general form

$$g(x) = -\frac{1+\theta_0(\theta_0 x - 1)}{\sqrt{(\theta_0 x-1)^2 + x}} \tag{38}$$

where $x = \lambda_p^2 \geq 0$. The derivative of the "minima" function

$$g'(x) = -\frac{(3\theta_0 - 1) - \theta_0^2 x}{2[(\theta_0 x - 1)^2 + x]^{\frac{3}{2}}} \tag{39}$$

gives the extremum point at

$$x = \frac{(3\theta_0 - 1)}{\theta_0^2} \geq 0, \tag{40}$$

which means $\theta_0 \geq \frac{1}{3}$.

One has thus to solve

$$\max_{\theta_0 \geq \frac{1}{3}} \left\{ -\frac{1+\theta_0(\theta_0\lambda_0^2 - 1)}{\sqrt{(\theta_0\lambda_0^2-1)^2+\lambda_0^2}} \right\} \tag{41}$$

where $\lambda_0$ is the zero of (36) for a choice of $\theta_0$. One can use for this purpose the mathematical software packages.

The significant local minima of the function $f$, computed by using MATLAB software for $\theta_0 \geq \frac{1}{3}$ are given in Table 1. Fig. 3 and Fig. 4 (the zoom in on $(0,\pi)$) show the graphical representations of $f(\lambda)$ for both cases $\theta_0 \in [\frac{1}{3}, \frac{4}{\pi^2}]$ and $\theta_0 > \frac{4}{\pi^2}$. One can observe that, as we previously have estimated, the values of minima decrease when $\theta_0$ increases. On the other hand, it can be seen that for any of these functions the smallest minimum is within the interval $(0,\pi)$. In order to solve (41) we have thus to find the maximum of these absolute minima on $(0,\pi)$. The curve of the absolute minima of $f(\lambda)$ with respect to $\theta_0$ is illustrated in Fig. 5; also, the significant values are in Table I. We conclude that for the case $\theta_0 \in (\frac{1}{3},\infty)$, the minimum we search for is $\overline{f}_{min} = -0.64968$ obtained for $\theta_0 = 0.38$.

Fig. 6 shows the curves of $f(\lambda)$ for $\theta_0 \in [0,\frac{1}{3})$ (the negative value of $\theta_0$ is taken from curiosity). These confirm that the absolute minimum is in this case $\lambda = 0 \in [0,\pi)$. But for this case we have already obtained that $f(0) = g(0) = \theta_0 - 1 \in [-1,-\frac{2}{3})$, thus the maximum "absolute minimum" for $\theta_0 \in [0,\frac{1}{3})$ is less than $-0.64968$, the value of the minimum for $\theta_0 \in (\frac{1}{3},\infty)$.

We conclude that the solution of the *minimax* problem (28) is

$$-0.64968 \geq -\frac{1}{k_0} \tag{42}$$

and thus we have determined the sector of absolute stability

$$k < \frac{1.5392}{\tau}. \tag{43}$$

The result of our analysis reads as follow

Consider the time-delay nonlinear system (21) with $\phi(\cdot)$ a continuous monotone increasing nonlinearity. The system is asymptotic stable for all nonlinear (and linear) functions verifying (20) and which are within the sector $(0, \frac{1.5392}{\tau})$.

*Remark 5.* Obviously, the absolute stability sector (43) is narrowed by increasing the time delay. Regarding the mathematical model (16) for the congestion problem in communication networks this time-delay dependent absolute stability condition means that the class of the nonlinear functions $\psi(\cdot)$ of the type (7) is diminished by increasing the round-trip time $\tau$.

## 3. CONCLUSIONS

This paper deals with the analysis of the absolute stability for a class of scalar nonlinear time-delay systems having monotone increasing nonlinearities. These systems can be encountered, for instance, as congestion problems in high-performance communication networks.

We have used an approach based on frequency domain inequalities of Popov type for time-delay systems in the critical case of the transfer function with a simple zero pole. In order to solve the *minmax* problem which arises from the frequency domain inequality, the analytical analysis was completed by numerical computations using MATLAB software package.

We have obtained a time-delay dependent absolute stability condition i.e. the class of the nonlinear functions, and therefore of the nonlinear systems under consideration, can be limited by large time-delays (for instance, the round-trip time in the case of the congestion problems in high-performance communication networks).

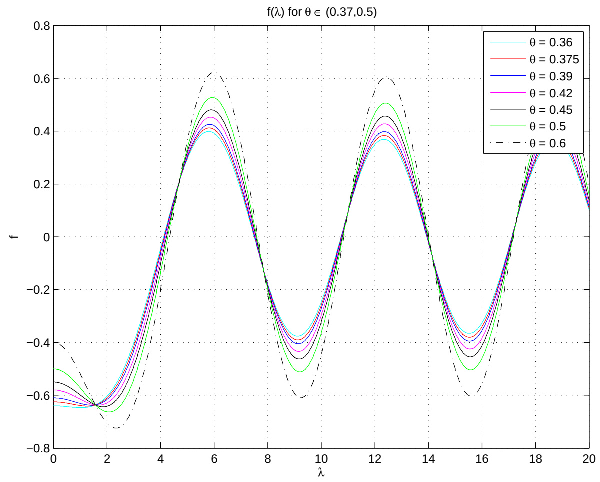


Fig. 3. Function $f$ for $\theta \in (\frac{1}{3}, \frac{1}{2})$.

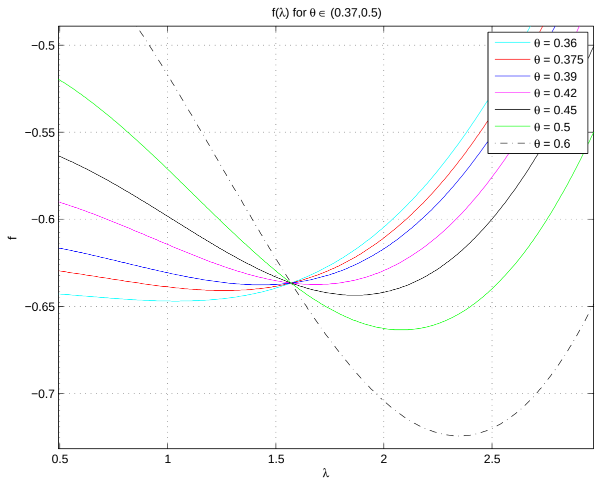


Fig. 4. Zoom in: function $f$ for $\lambda \in (0, \pi)$ and $\theta \in (\frac{1}{3}, \frac{1}{2})$.

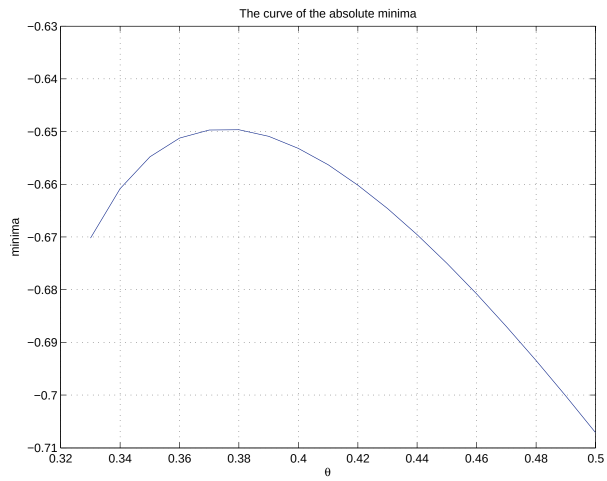


Fig. 5. The absolute minima of $f(\lambda)$ for $\theta \in (\frac{1}{3}, \frac{1}{2})$.

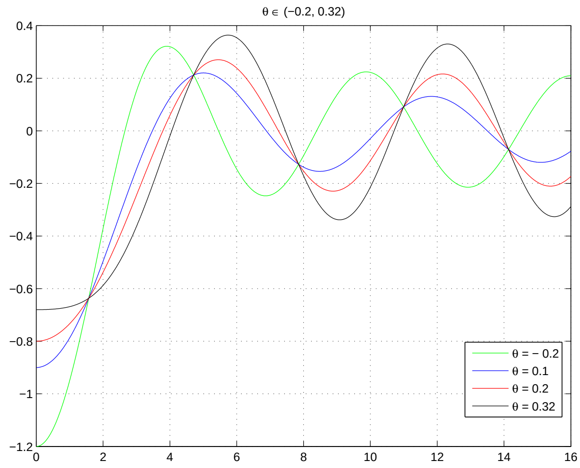


Fig. 6. Function $f$ for $\theta \in (-0.2, \frac{1}{3},)$.

Table 1. The maximum minima of $f(\lambda)$ for $\theta \in (\frac{1}{3}, \frac{4}{\pi^2})$

| $\theta_0$ | $f(\lambda_0)$ |
|---|---|
| 0.33 | -0.6702 |
| 0.34 | -0.6608 |
| 0.35 | -0.6548 |
| 0.36 | -0.6513 |
| 0.37 | -0.6497 |
| 0.38 | -0.64968 |
| 0.39 | -0.6510 |
| 0.40 | -0.6532 |
| 0.41 | -0.6563 |
| 0.42 | -0.6602 |

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