# ONE-DIMENSIONAL REPRESENTATIONS OF DYNAMICAL SYSTEMS AND APPLICATIONS TO CONTROL

**Xu Xu, Yi Song**

*Department of Automatic Control and Systems Engineering,*
*University of Sheffield,*
*Mappin Street,*
*Sheffield, S1 3JD.*
*e-mail: cop06xx@sheffield.ac.uk*

Abstract: We shall demonstrate a method for reducing n-dimensional dynamical systems to 1-dimensional ones by using a symbolic, 2-adic representation of the states. The dynamics will then appear as a 1-dimensional graph and periodic points and other invariant sets can be found. Applications to the evaluation of switching surfaces in bang-bang control will be given.

Keywords: Dynamical systems; One-dimensional representations; Bang-bang control.

## 1. INTRODUCTION

In this paper, we will show how to obtain a one-dimensional representation of an n-dimensional dynamical system, which captures the essential behaviour of the system, such as periodic solutions, invariant sets, etc. in a visually simple form. Each point in the $n-D$ space has a unique representation in the corresponding $1-D$ space if we remove certain points from the $n-D$ space. Hence, all information of the dynamical system will be contained in a graph which is easier to observe. To illustrate the application of the method to control these we shall consider the problem of time-optimal control, and show that it makes the generation and visualisation of the switching manifolds relatively easy.

We shall present a simple way of finding the switching curve/surface of time-optimal Bang-bang control in Section 3. And then two examples in 2-dimensional and 3-dimensional cases will be given in Section 4.

## 2. ONE-DIMENSIONAL REPRESENTATIONS OF N-DIMENSIONAL SYSTEMS

The basic idea consists of using a symbolic, 2-adic representation of the states of an n-dimensional dynamical system

$$\dot{x} = f(x) \qquad (1)$$

where $f : E \to \Re^n$ and $E$ is an open subset of $\Re^n$, to get a 1-dimensional map.

To do this, we first normalise the dynamical system. Put

$$x_i = \frac{1}{\pi}[\arctan(\alpha y_i)] + \frac{1}{2}, \quad i = 1, 2, \cdots, n$$

which maps $\Re^n$ into $I^N$, where $I = [0, 1]$ and $I^N = I \times \cdots \times I$.

In a 2-dimensional system's case, we can map $I^2$ to $I$ in the following way:

$$(x_1, x_2) \in I^2 \longrightarrow \widetilde{x_1 x_2} \in I$$

where $\widetilde{x_1 x_2}$ is given as follows:
Write $x_1, x_2$ in their binary forms

$$x_1 = 0.b_1 b_2 b_3 \cdots\cdots$$
$$x_2 = 0.c_1 c_2 c_3 \cdots\cdots$$

and define

$$\widetilde{x_1 x_2} = 0.b_1 c_1 b_2 c_2 b_3 c_3 \cdots\cdots ,$$

we get a map $\nu : I^2 \longrightarrow I$, $\nu(x_1, x_2) = \widetilde{x_1 x_2}$.

If we remove all points ending with infinite number of 1$s$, i.e. $\cdots 1111111 \cdots$ for both $x_1$ and $x_2$ in the $2 - D$ plane, in other words, we chop out all double-valued points (Xu, Song and Banks), then all other points have unique representations on the $1 - D$ system according to the map $\tilde{\nu}$:

$$I^2 \backslash S \xrightarrow{\tilde{\nu}} I$$

where $S$ is the set of all double-valued points in $2 - D$ plane which is a countable subset $S \subseteq I^2$.

Conversely, by removing all points of the form $\cdots 1 \bullet 1 \bullet 1 \bullet 1 \bullet 1 \cdots$ on the $1 - D$ plane, we got a unique correspondence on $2 - D$ plane of all remaining points. Since those points are not all double-valued and the set $U$ of them is certainly uncountable, not every point on $1 - D$ plane has representation on $2 - D$ plane, i.e. there is a $1 - 1$ map $\tilde{\tilde{\nu}}$:

$$I^2 \backslash S \xrightarrow{\tilde{\tilde{\nu}}} I \backslash U$$

where $U \subseteq I$.

Since $\Re^2$ and $\Re$ are topologically inequivalent (Crutchfield, 1994), the phenomenon above can be explained by the discontinuity of $\tilde{\tilde{\nu}}$, so that we see two points which are arbitrarily close in $I^2 \backslash S$ can be far away in $I \backslash U$.

If we discretise a dynamical system in the form of (1) (but just for 2-dimensional case) to be

$$x_k = F_h(x_{k-1}) \tag{2}$$

where $x_k = (x_{k1}, x_{k2})$, and $h$ is the step-length, then the map $f$ between two states $X_0$ and $X_1$ in the original system becomes a map $F_h$ from $B_0$ to $B_1$, where $B_0, B_1 \in I^2 \backslash S$.
Let $\tilde{N}$ be a nonlinear map from $D_0$ to $D_1$, determined by $F_h$ in $2 - D$ space, where $D_0$, $D_1 \in I \backslash U$, which makes the diagram

$$\begin{array}{ccc} B_0 & \xrightarrow{\tilde{\tilde{\nu}}} & D_0 \\ {\scriptstyle F_h}\downarrow & & \downarrow{\scriptstyle \tilde{N}} \\ B_1 & \xrightarrow{\tilde{\tilde{\nu}}} & D_1 \end{array}$$

commutative.

By applying the map $\tilde{N}$ $q$ times, which gives the function $\tilde{N}^q$, we have a graph showing some interesting behaviours of the dynamical system, such as equilibrium points and periodic orbits (Xu, Song and Banks).

A picture of the function $\tilde{N}^q$ (also depends on the step length $h$, of course) of Van-der-Pol oscillator is shown below:

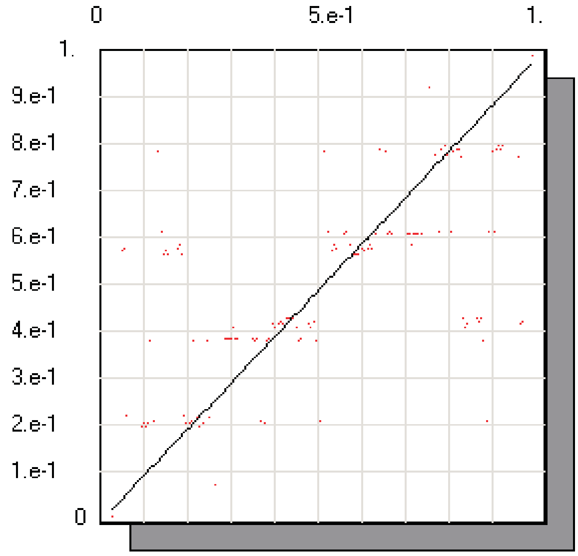


Fig.1 $\tilde{N}^q$ of Van-der-Pol oscillator

Points on the diagonal of the graph of $\tilde{N}^q$ clearly represent the periodic points in the original dynamical system. We can also get an iterated picture starting from an initial point which is on the periodic orbit, showing the periodic trajectory mapped into $1 - D$ space (Lind and Marcus, 1995).

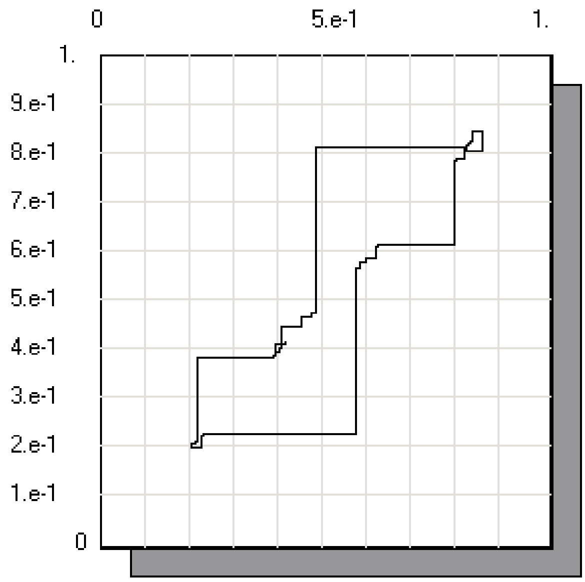


Fig.2 A periodic orbit

For simplicity, we only considered the $2-D$ case above, however, the same method applies for $n-D$ dynamical systems.

*Theorem 1.* There exists a countable subset $S \subseteq I^N$ and an uncountable subset $U \subseteq I$ such that the induced map $\tilde{\tilde{\nu}}$:

$$I^N \backslash S \longrightarrow I \backslash U$$

is $1-1$.

For instance, we can obtain a iterated graph showing the chaotic behaviour starting from a random initial point of Lorenz-Attractor.

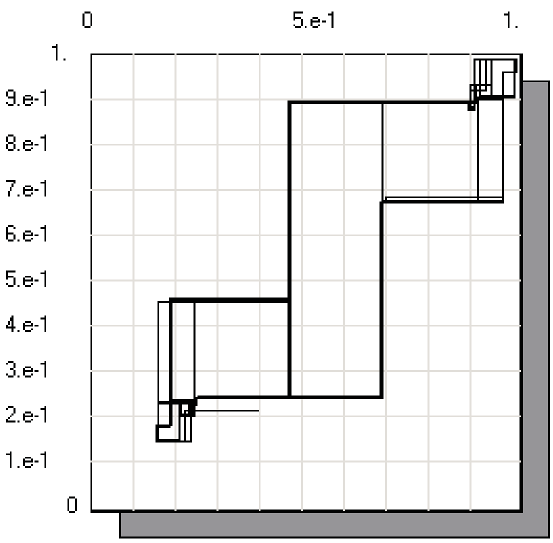


Fig.3 Iterated graph of Lorenz-Attractor

## 3. THE TIME-OPTIMAL CONTROL PROBLEM

It is well-known (see (Pontryagin, 1962)) that for a linear system

$$\dot{x} = Ax + Bu \tag{3}$$

with hard constraint $|u| \leq 1$, the time-optimal problem has solutions with at most $n-1$ switches. In high dimensions the switching manifolds are difficult to visualise. Hence we apply the above technique to plot the switching manifolds in 1-dimensional spaces.

For a point on the switching line, no switch is necessary. Simply use the appropriate control to get to the origin. For any other point, following the control curve first until it hits the switching line (the opposite type of control), then switch to the second one and the solution goes to the origin. In order to reach the origin in minimum time, we use the least number of switches. Obviously, in a $2-D$ Bang-bang control system, only 1 switch is needed at most. Hence the problem becomes how to determine the switching curve in the phase plane and decide when to switch.
The situation for $3-D$ systems is a bit more complicated, but we can still reduce the problem into finding the switching curve and also the switching surface.

It is not so easy to achieve the positions of switching manifolds by using traditional ways, especially when the dimensions of the systems increase. Nonetheless, if we map the $n-D$ system into a $1-D$ plane introduced before, there appears a applicable way of solving this difficult problem.

*Theorem 2.* To determine the control of a linear dynamical system

$$\dot{x} = f(x, u)$$

with $f \in C^1(E)$ where $E$ is an open subset of $\Re^n$, $n$ $1-D$ space graphs are needed. The first graph always indicates the location of the switching line; the $(n-1)th$ one shows exactly the $n-1$ Dimensional Switching Manifold of the control system and the initial control status is contained in the $nth$ plot.

## 4. EXAMPLES

In this section we shall give two simple examples of Bang-bang time-optimal control systems. By applying the idea illustrated in Sections 2 and 3, plots containing information of switching curve, switching surface, and also first control state can be easily obtained.

**Example1.** For a 2-dimensional Bang-bang control system of the form:

$$\dot{x_1} = 2x_2$$
$$\dot{x_2} = u = \pm 1$$

we get two plots shown in fig 4 below.

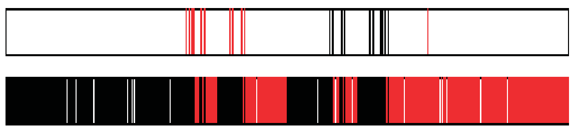

Fig.4

Red bars in the first plot correspond to points in half of the switching line using $-1$ control on the original $2-D$ plane; and black bars represent points in the other half ($+1$ control). In the second plot, points in the red region can reach the origin by using $-1$ control first, then switching into $+1$ control when hit the switching line. Conversely, points in the black area will need $+1$ then $-1$ controls to get to the origin.

**Example2.** Similarly, in a 3-dimensional Bang-bang control system

$$\dot{x_1} = x_2$$
$$\dot{x_2} = x_3$$
$$\dot{x_3} = u = \pm 1$$

three graphs can be got in fig5.

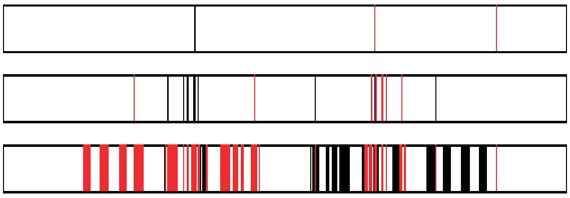

Fig.5

Again, red and black bars in graph 1 denote the switching curve in the $3-D$ space while for these points, no switch is needed; red and black regions in the second graph suggest the switching surface in the original space and only one switch of control is necessary for points here; the third graph presents initial control values for all other points which need two switches.

## 5. CONCLUSIONS

We have demonstrated a method for encoding solutions of an n-dimensional system in a one-dimensional space by a certain map $\tilde{\tilde{\nu}}$. We have also shown how to use this idea to visualise the switching manifolds of linear time-optimal control systems. In a future paper we shall study the continuity property of the map $\tilde{\tilde{\nu}}$ and also apply this idea in controlling higher-dimensional nonlinear systems.

## REFERENCES


Crutchfield J. P.(1994). The Calculi of Emergence: Computation, Dynamics and Induction. In: *Physica D*, 75, 11-54.

Lind D., and Marcus B.(1995) *Symbolic Dynamics and Coding*, CUP: Cambridge.

Pontryagin L. S., Boltyanski V. A.,Gamkrelidze R. V. , and Mischenko E. F.,(1962) *The Mathematical Theory of Optimal Processes*, Wiley, New York.

Xu X., Song Y. and Banks S. P. (2007), On the Dynamical Behaviour of Cellular Automata. Submitted to *Journal of Mathematical Analysis and Applications*.

Xu X., Song Y. and Banks S. P.(2007), Cellular Automata, Symbolic Dynamics and Artificial Life. In: *Internatioanl Carpathian Control Conference ICCC'2007*, p.776-779.