FROM LOCAL TO GLOBAL CONTROL SYSTEMS

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Abstract: In this paper, we first consider dynamical systems by applying a decomposition to the global manifold and control the local representations. Then we investigate the problem conversely, i.e., given a set of local systems, we shall consider determining the dynamics and topology of the global manifold.

Keywords: dynamical systems, global decomposition, tangent bundle, transition map.

1. INTRODUCTION

The theory of dynamical systems on compact manifolds is well known (see, e.g., (Isidori, 2005)), and the study of such systems can be reduced to a finite number of local problems of the form

$$\dot{x} = f(x), \tag{1}$$

where each of them is defined on some neighbourhood of the global manifold M. The control of the systems can be realized by applying the standard technique within each region and then combine them together. Also in many cases, we have the opposite situation – we are given several local systems (defined around 'operating points') and we wish to determine the global structure of the dynamics and the topology of the underlying manifold. In this paper, we will consider control systems in both ways, i.e., from global to local, and from local to global.

2. FROM GLOBAL TO LOCAL CONTROL SYSTEMS

A dynamical system on an n-manifold M, is defined by a global section of the tangent bundle TM. We shall assume that M is compact, so that it has a finite number of coverings each consisting of a local neighbourhood and chart (u_i, ϕ_i), such that $u_i \subset M$ and $\phi : u_i \to \phi_i(u)$ is a homeomorphism, where $\phi(u_i)$ is an open set in $\mathbb{R}^n$. So locally in terms of each chart the system takes the form of (1). Hence, given (u_1, ϕ_1) as a local coordinate neighbourhood on M (i.e., $u_1 \subset M$ is open and $\phi_1 : u_1 \to \mathbb{R}^n$ is a homeomorphism), locally the system can be written as

$$\dot{x} = f_1(x), \qquad x \in \phi_1(u_1) \subseteq \mathbb{R}^n. \tag{2}$$

If there exists another local neighbourhood (u_2, ϕ_2) such that $u_1 \cap u_2 \neq \varnothing$, the system takes the form

$$\dot{y} = f_2(y), \qquad y \in \phi_2(u_2) \subseteq \mathbb{R}_n. \tag{3}$$

then the relation between these two local systems is given by:

$$\frac{\partial y}{\partial x} f_1(x) = f_2\big(y(x)\big) \tag{4}$$

where $y(x) = \psi_{12}(x)$, and ψ_{12} is the transition function from u_1 and u_2, i.e., $\psi_{12} : u_1 \to u_2$. Note that generally ψ satisfies the 'cocycle condition', i.e.,

$$\psi_{\alpha\beta} \cdot \psi_{\beta\gamma} = \psi_{\alpha\gamma} \qquad \text{on } u_\alpha \cap u_\beta \cap u_\gamma. \tag{5}$$

Hence it implies

$$\begin{cases} \psi_{\alpha\alpha} \cdot \psi_{\alpha\alpha} = \psi_{\alpha\alpha} = I \text{ on } u_\alpha \\ \psi_{\alpha\beta} \cdot \psi_{\beta\alpha} = \psi_{\alpha\alpha} = I \text{ on } u_\alpha \cap u_\beta \end{cases} \tag{6}$$

Moreover, the maps $\phi_{\beta\alpha} = (\partial\psi_{\beta\alpha})/\partial x$ are the transition maps for the tangent bundle.

In (Song, Xu and Banks), we proposed an optimal tracking scheme of dynamical systems which are situated on compact manifolds by using the above global decomposition method. Explicitly speaking, we first divide the whole manifold into a finite number of individual regions such that one region will intersect with some other regions, then applying the iteration technique (introduced in (Banks and Dinesh, 2000)) to each local representation of the system gives rise to a local optimal control, and the global result will be achieved by combining them together via the link of the transition functions $\psi_{\alpha\beta}$ (see fig.(1) for illustration).

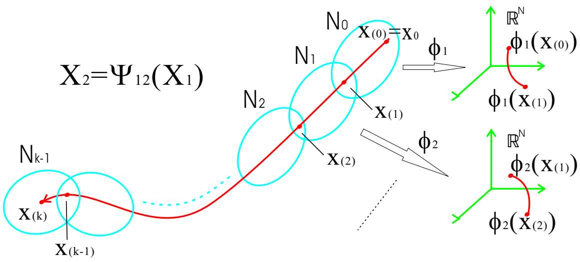


Fig. 1. Local control give rise to global result

For example, an aeroplane flight can be defined on a compact manifold M, where several states are involved: take off, climbing, cruising, descending and landing. We can apply the global decomposition to M, such that each of the states has a local representation defined on some neighbourhood of the global manifold, intersecting with the region that the next state is situated. When the plane is working in a specific region, say climbing, we switch to the local representation of the system in that region and apply the local control around some 'trim' condition to achieve the local objective. Once it enters the intersecting region, we can then make the smooth change towards the next state, cruising in this case, by applying the transition function. A new control is needed to be applied according to the specific state.

3. FROM LOCAL TO GLOBAL CONTROL SYSTEMS

Conversely, given a finite number of local systems,

$$\begin{cases} \dot{x} = f_1(x) \\ \dot{y} = f_2(y) \\ \quad \vdots \\ \dot{z} = f_k(z) \end{cases} \tag{7}$$

where $x, y, z \in \mathbb{R}^n$, and there exist a homeomorphism $\phi^{-1} : \mathbb{R}^n \to u_i$ $(u_i \in \mathbb{R}^N, N \geq n)$.

Assume that u_i $(1 \leq i \leq k)$ is a complete cover of some manifold M, i.e.,

$$\bigcup_{i=1}^{k} u_i = M,$$

and if $u_i \cap u_j \neq \emptyset$ $(1 \leq i, j \leq k)$ then within u_{ij}, the dynamical structure (i.e., number of equilibria, sectors around them, etc.) on u_i and u_j are the same.

From previous chapter, we know that for a set of local systems to be able to form a combinatorial system on some manifold, there must be one transition map corresponding to each intersecting region, e.g., given

$$\begin{cases} \dot{x} = f(x) \\ \dot{y} = g(y) \end{cases} \tag{8}$$

where $\phi_i^{-1}(x) \subseteq u_i$ and $\phi_j^{-1}(y) \subseteq u_j$, there exists a homeomorphism ψ such that $y = \psi_{ij}(x)$ and the following diagram commutes

$$\begin{array}{ccc} T(\mathbb{R}^n) & \xrightarrow{\frac{\partial\psi}{\partial x}} & T(\mathbb{R}^n) \\ f \uparrow & & \uparrow g \\ \mathbb{R}^n & \xrightarrow{\psi} & \mathbb{R}^n \end{array}$$

Hence

$$g\big(\psi_{ij}(x)\big) = \frac{\partial\psi_{ij}}{\partial x} \cdot f(x). \tag{9}$$

This is a PDE for ψ_{ij}, it must have a solution. Furthermore, these transition functions ψ must be compatible in the sense that if $u_i \cap u_j \cap u_l \neq \emptyset$ (as shown in *fig.*), then

$$\frac{\partial\psi_{j\,l}}{\partial y} \cdot \frac{\partial\psi_{ij}}{\partial x} = \frac{\partial\psi_{il}}{\partial x} \tag{10}$$

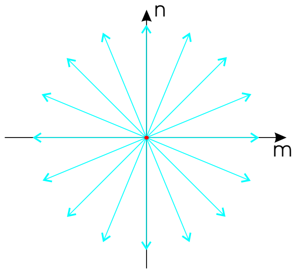
(since

$$\frac{\partial z}{\partial x}\cdot f(x)=\frac{\partial \psi_{j\,l}}{\partial y}\cdot\frac{\partial \psi_{ij}}{\partial x}\cdot f(x)=\frac{\partial \psi_{il}}{\partial x}\cdot f(x).)$$

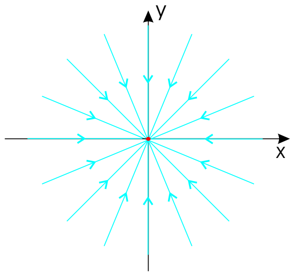
In this way, we can obtain the combinatorial dynamics by the transition mappings and determine the topology of the global manifold M. Note that the resulting index of M must add up to satisfy the *Poincaré-Hopf* theorem.

Example 1. Given two 2-dimensional local systems $\big($see fig.(2)$\big)$,

$$\begin{cases}\dot m = m\\ \dot n = n\end{cases}\text{ and }\begin{cases}\dot x=-x\\ \dot y=-y\end{cases}\tag{11}$$



(a) Unstable node



(b) Stable node

Fig. 2. Phase plane dynamics

where

$$\begin{cases}m=f(x,y)\\ n=g(x,y)\end{cases},$$

so we have

$$\begin{cases}\dot m=m=\dfrac{\partial f}{\partial x}\cdot(-x)+\dfrac{\partial f}{\partial y}\cdot(-y)\\ \dot n=n=\dfrac{\partial g}{\partial x}\cdot(-x)+\dfrac{\partial g}{\partial y}\cdot(-y)\end{cases}\tag{12}$$

One possible solutions for (12) is

$$\begin{cases}m &= \dfrac{x}{x^2+y^2}\\ n &= -\dfrac{y}{x^2+y^2}\end{cases}.$$

By using the complex representation, we have

$$\begin{cases}z &= x+iy\\ z' &= m+in\end{cases},$$

and the transition function will be

$$z'=\psi(z)=\frac{1}{z}.\tag{13}$$

It is obvious to see that the global dynamics given by (11) is defined on S^2 $\big($as shown in fig.(3)$\big)$, and the total index add up to the *Euler* Characteristic of a 2-sphere, i.e., 2.

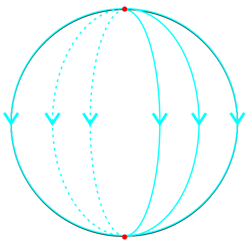
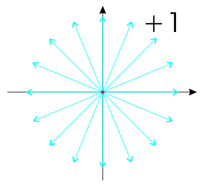


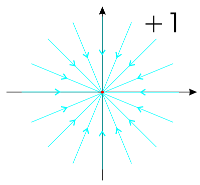
Fig. 3. Global dynamics defined on S^2

Note that by choosing different intersecting regions will yield different transition functions, hence give rise to different global manifolds.

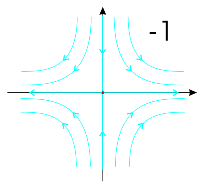
Specifically speaking, in two-dimensional case, we can check this by observing the total index which equals to the *Euler* Characteristics, an invariant that can distinguish all 2-manifolds. For example, given six local regions with dynamics defined on one each (see fig.(4) for illustration, 1 stable node, 1 unstable node and 4 saddle nodes),



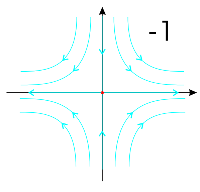
(a)



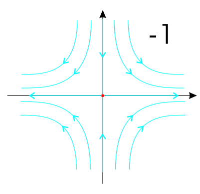
(b)



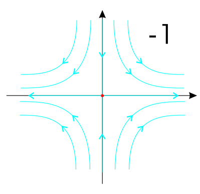
(c)



(d)



(e)



(f)

Fig. 4. Individual dynamics of six local regions

If we take all the intersecting regions to be away from the equilibria, we then get a combinatorial system with index -2, hence it is possible for the global dynamics to be situated on a genus-2 torus. Meanwhile, we can take the equilibria of fig.(4c) to (4f) into the intersecting regions and identify them pairwise, i.e., fig.(4c) with fig.(4d), fig.(4e) with fig.(4f). In this way, the resulting system will have 0 index, which shows that the global structure can be situated on a torus.

In 3-manifolds, the same checking algorithm for 2-manifolds doesn't work due to the fact that the *Euler* characteristics for all 3-manifolds is 0. However, because we can obtain different 3-manifolds by simply adding a Dehn twist to the transition functions, the same local systems can possibly have their global dynamics sits on different 3-manifolds.

In 4-manifolds case, we can regard the local systems as tangent bundles attached to some manifold and hence study the connections and curvatures of these bundles. If within the intersecting region, u_{ij}, the Characteristic classes for u_i and u_j match, and the argument is valid for all the intersections, then globally, the systems fit together on some 4-manifold, of which to study the topology will involve sophisticated invariants, e.g., Seiberg-Witten invariants. Also, the uniqueness of the global manifold can not be guaranteed.

Example 2. We now consider dynamical systems over S^4 (4-sphere), regarded as the one-point compactification of the quaternion numbers, $S^4 = Q \cup \{\infty\}$. Give Q the standard quaternion coordinate q and let $U_0 = S^4 - \{\infty\}$, $U_\infty = S^4 - \{0\}$ be the two local neighbourhoods. Since the dynamical systems are given in terms of tangent bundle, the transition map is generated by (see, e.g. (Moore, 1996))

$$g_{\infty\,0}(q) = \frac{1}{q^2}, \quad \text{on} \quad U_0 \cap U_\infty \tag{14}$$

Given two 4-dimensional local systems, a stable node and an unstable node,

$$\begin{cases} \dot{x} = x \\ \dot{y} = y \\ \dot{z} = z \\ \dot{r} = r \end{cases} \text{and} \begin{cases} \dot{x} = -x \\ \dot{y} = -y \\ \dot{z} = -z \\ \dot{r} = -r \end{cases}. \tag{15}$$

defined on U_0 and U_∞ respectively. Using quaternion coordinate,

$$q = x + iy + jz + kr, \quad \text{and}$$

we have

$$\begin{aligned} &\dot{q} = q(\text{stable node}) \\ \& \quad &\dot{q} = -q(\text{unstable node}). \end{aligned}$$

Now consider the possibility of these two local systems to be situated on S^4 (4-sphere).

Using the matrix representation, the two local sections are

$$\sigma_{\mathbf{o}} = \begin{pmatrix} x & -y & -z & -r \\ y & x & -r & z \\ z & r & x & -y \\ r & -z & y & x \end{pmatrix} \tag{16}$$

$$\sigma_\infty = \begin{pmatrix} -x & y & z & r \\ -y & -x & r & -z \\ -z & -r & -x & y \\ -r & z & -y & -x \end{pmatrix}. \tag{17}$$

Given a 4×4 matrix,

$$\omega_0 = dq = \begin{pmatrix} dx & -dy & -dz & -dr \\ dy & dx & -dr & dz \\ dz & dr & dx & -dy \\ dr & -dz & dy & dx \end{pmatrix}, \tag{18}$$

since the connection d_A is defined by

$$d_A\sigma = d\sigma + \omega\sigma, \tag{19}$$

we have, for σ_0

$$\begin{aligned} d_{A_0} = &\begin{pmatrix} dx & -dy & -dz & -dr \\ dy & dx & -dr & dz \\ dz & dr & dx & -dy \\ dr & -dz & dy & dx \end{pmatrix} \\ &+\omega_0 \cdot \begin{pmatrix} x & -y & -z & -r \\ y & x & -r & z \\ z & r & x & -y \\ r & -z & y & x \end{pmatrix} \end{aligned}$$

Also, the curvature of a connection is given by

$$\Omega(\sigma) = (d\omega + \omega \wedge \omega)\sigma, \tag{20}$$

hence, for σ_0, we have

$$\Omega_0 = d \begin{pmatrix} dx & -dy & -dz & -dr \\ dy & dx & -dr & dz \\ dz & dr & dx & -dy \\ dr & -dz & dy & dx \end{pmatrix} + \omega_0 \wedge \omega_0.$$

Since for 1-form ω_0, $\omega_0 \wedge \omega_0 = 0$, and $d(dx) = d(dy) = d(dz) = d(dr) = 0$ by definition, substitute (16) into (20), we prove Ω_0 is skew-Hermitian. Moreover,

$$\text{Trace}\left[\left(\frac{i}{2\pi}\Omega_0\right)^k\right] = 0. \tag{21}$$

For section σ_∞, the 4×4 matrix ω_∞ is given by

$$\begin{aligned} \omega_\infty &= g_{\infty\,0} d(g_{\infty\,0}^{-1}) + g_{\infty\,0}\omega_0 g_{\infty\,0}^{-1} \\ &= \frac{2}{q} \cdot dq + dq \end{aligned}$$

where

$$\frac{2}{q} = \begin{pmatrix} \frac{2x}{|q|^2} & \frac{2y}{|q|^2} & \frac{2z}{|q|^2} & \frac{2r}{|q|^2} \\ -\frac{2y}{|q|^2} & \frac{2x}{|q|^2} & \frac{2r}{|q|^2} & -\frac{2z}{|q|^2} \\ -\frac{2z}{|q|^2} & -\frac{2r}{|q|^2} & \frac{2x}{|q|^2} & \frac{2y}{|q|^2} \\ -\frac{2r}{|q|^2} & \frac{2z}{|q|^2} & -\frac{2y}{|q|^2} & \frac{2x}{|q|^2} \end{pmatrix}, \qquad (22)$$

and $|q|^2 = x^2 + y^2 + z^2 + r^2$. The diagonal entries for ω_∞ are all

$$\omega_{\infty_{ii}} = \left(\frac{2x}{|q|^2} + 1\right)dx + \frac{2y}{|q|^2}dy + \frac{2z}{|q|^2}dz + \frac{2r}{|q|^2}dr. \qquad (23)$$

Substitute ω_∞ into (19) and (20), we then get the connection d_{A_∞} and curvature Ω_∞ for section σ_∞, respectively. Likewise, Ω_∞ is skew-Hermitian, and from (23), we obtain

$$\text{Trace}\left[\left(\frac{i}{2\pi}\Omega_\infty\right)^k\right] = 0. \qquad (24)$$

Compare (21) and (24), we have

$$\text{Trace}\left[\left(\frac{i}{2\pi}\Omega_0\right)^k\right] = \text{Trace}\left[\left(\frac{i}{2\pi}\Omega_\infty\right)^k\right.$$

on $U_0 \cap U_\infty$. Hence these locally defined forms fit together into a globally defined real-valued $2k$-form $\tau_k(A)$, where $\tau_k(A)$ is a characteristic form.

So the two locally defined systems given by (15) can fit together to be situated on S^4. Generally speaking, a local stable node and a local unstable node that are both n-dimensional can always be situated on S^n.

4. CONCLUSION

In this paper, we first study the optimal control of a global system by splitting it into a finite number of local systems and applying local optimizations individually. Then we mainly concentrated on considering the possibility of combining a given number of locally defined systems to achieve a global result by studying the topology of the global manifold. In the future we will further this research by using Chern classes and Seiberg-Witten invariants.

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