NETWORKS AND SYNCHRONIZATION

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Abstract: This paper presents some network models in connection to basic qualitative dichotomies: stability/instability and synchronization/chaos. Another dichotomy deals with interconnections which may be delayed or diffusive. A special model leading to time delay partial differential equations is presented.

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1. INTRODUCTION. MODELS AND PROBLEMS

A. Speaking or writing about networks, regardless their physical nature, means representation of some "devices" or sub-systems subject to some interconnections of definite type resulting in a system with new properties. The systemic approach to such structures means that each sub-system is viewed as some processor of the input signals which are transformed in output signals sent to other sub-systems. If we forget the purely physical aspects then system's evolution in time becomes an information processing.

Among the simplest dynamical systems we have to mention those with a single globally asymptotically stable attractor: this attractor may be an equilibrium or a periodic motion (maybe almost periodic). The next class of systems from the complexity point of view is that of the systems with several asymptotically stable attractors. The best known are here the *neural networks*: their capacity as devices of the Natural/Artificial Intelligence is in direct connection with the number of the attractors. This systems classification allows emphasizing some dichotomic types of qualitative behavior: stability/instability and synchronization/chaotic behavior. This behavior is related to interconnections: they may send the system to the stable equilibrium or "push" it to instability. Also they may send to order and synchronous behavior of sub-systems' motions or, in the opposite case, to complex, chaotic evolutions.

Since the Liapunov stability is well understood, including systems with several equilibria - see Gelig et al (1978), Leonov et al (1992), Halanay and Răsvan (1993), we shall deal with the second dichotomy, especially with synchronization.

B. Synchronization means that the interconnections have as direct consequence some regularity of the sub-systems' behavior. The most obvious one is the complete synchronization of the motion implying the periodicity of all state variables of the system. Such type of behavior is quite known - synchronization of the mechanical vibrators, of the electrical machinery, even of the biological processes where synchronization is viewed as a mechanism of self-organization.

The mathematical model of the synchronization in the state space is the existence of an invariant toric manifold composed of quasi-periodic motions. The synchronisms are classified according to their degree. The highest synchronization degree corresponds to a synchronous motion of all subsystems, with the same period; this corresponds to a toric manifold of minimal dimension (=1) i.e.

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to a periodic motion. The lowest synchronization degree corresponds to the independent periodic motions of all sub-systems, each with its own period; this corresponds to a maximal dimension of the toric manifold.

C. The best known synchronization models are the phase synchronization. It is interesting to remark a certain similarity of all these models. For instance, the paradigm-model of Kuramoto - see Kuramoto (1975), Kuramoto (1984) - with the form

$$\begin{aligned}&\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_1^N \sin\,(\theta_j - \theta_i) + \xi_i(t),\\ &i = 1, \ldots, N\end{aligned} \tag{1}$$

where θ_i is the oscillator's phase, ω_i - its natural frequency and $\xi_i(t)$ - a perturbation accounting for the uncertainty, is very similar to the model of R. Adler - the inventer of the remote control - see Adler (1946), Acebrón and Spigler (2007)

$$\dot{\alpha} = \omega_0 - \omega_1 - \frac{E_1}{E}\,\frac{\omega_0}{2Q}\sin\,\alpha \tag{2}$$

This models concerns a single oscillator, where (ω_1, E_1) are the electrical parameters of the external signal, ω_0 - the current frequency and α - the phase of the oscillator.

The model of the oscillator in the micro-wave networks is of the same type (Adler)

$$\dot{\theta} = \omega_0 + \frac{\omega_0}{2Q}\,\frac{A_i(t)}{A(t)}\sin\,(\theta_i(t) - \theta) \tag{3}$$

with $(A_i(t), \theta_i(t))$ - the amplitude and the phase of the external signal; the frequency is here constant. If we consider now N Adler oscillators that are synchronized to the frequency of the first one, the following model is obtained

$$\dot{\theta}_i = \omega_i - \frac{\omega_1}{2Q}\sum_{j=1}^N \varepsilon_{ij}\frac{\alpha_j}{\alpha}\sin\,(\psi_{ij} + \theta_i - \theta_j) \tag{4}$$

with ψ_{ij} - "added" phases; the model is very similar to that of Kuramoto. In these models the types of connections among individual oscillators are not specified: usually these are connections to some "neighbors" - the "closest" - and these couplings are represented by ε_{ij}.

It is interesting to consider another comparison : the models of Adler and Kuramoto *versus* the electromechanical model of $n+1$ synchronous machines of power system analysis - see e.g. Vayman (1981)

$$\begin{aligned}&\dot{\theta}_i = \omega_i\\ &H_i\dot{\omega}_i = P_{mi} - \sum_{j\neq i}^{n+1} E_iE_jY_{ij}\sin\,(\theta_i - \theta_j) - \\ &-D(\omega_i)\ ,\ i = 1, \ldots, n+1\end{aligned} \tag{5}$$

This model contains also a dynamics of the frequencies but the couplings are as in the model of Adler and Kuramoto. Besides this the model (5) can represent also the pendula coupled by springs having sinusoidal characteristics Starjinskii (1985).

A simplified model of p.c.m. telephone networks reads Parks and Miller (1970)

$$\begin{aligned}&\dot{\phi}_{ij} = f_{0j} - f_{0i} - \sum_{l\neq i,j}[k_{jl}f(\phi_{jl}) - k_{il}f(\phi_{il})]\\ &i, j = 1, \ldots, n\end{aligned} \tag{6}$$

where

$$f(\sigma) = \sigma + [1/2 - \sigma]_e$$

is a N-periodic function, $[\ldots]_e$ being as usual the entire part of a number. This model is also of Adler type.

2. INTERCONNECTIONS AND DELAYS

A. An assumption that is supposed to make the network models more realistic is signal delay along the connection lines. A first possibility is the introduction of a "uniform" delay - the same for all interconnection lines - in the Kuramoto model with a unique frequency of all oscillators, see the paper of Earl and Strogatz (2003)

$$\dot{\theta}_i = \omega + \frac{K}{k}\sum_1^N a_{ij}f(\theta_j(t-\tau) - \theta_i(t)) \tag{7}$$

where each oscillator receives k signals and a_{ij} account for the presence or absence of some connection: $a_{ij} = 1$ shows that the oscillator j sends a synchronization signal to the oscillator i; in the opposite case $a_{ij} = 0$; also $\sum_j a_{ij} = k,\ \forall i$.

A more complicated model supposes separate delays in the classical Kuramoto model, see Papachristodoulou and Jadbabaie (2005)

$$\begin{aligned}&\dot{\theta}_i = \omega_i +\\ &+\frac{K}{N}\sum_1^N A_{ij}\sin\,(\theta_j(t-\tau_{ij}) - \theta_i(t))\end{aligned} \tag{8}$$

The telephone network model of Parks and Miller (1970) also contains delays over the transmission lines

$$\dot{\phi}_{ij} = f_{0j} - f_{0i} - \\ - \sum_{l \neq i,j} [k_{jl} f(\phi_{jl}(t - d_{ij})) - k_{il} f(\phi_{il}(t))] \ , \quad (9) \\ i,j = 1, \dots, n$$

In the modelling of the networks of biological oscillators the following model is considered Fox et al (2001)

$$\begin{aligned} \dot{x}_i &= 3x_i - x_i^3 - y_i + S_i \\ \dot{y}_i &= \varepsilon(f(x_i) - y_i) \\ S_i &= \sum_k w_{ik} S_\infty(x_k(t-\tau)) \\ S_\infty(x) &\equiv \left(1 - e^{-\alpha(x-\theta)}\right)^{-1} \end{aligned} \quad (10)$$

where θ is the threshold of the connecting signals.

B. A more interesting electrical model, where the delay arises from local propagation (it is present in each oscillator model) is that of Wu and Xia (1997)

$$\begin{aligned} & L_s \frac{\partial i_k}{\partial t} = -\frac{\partial v_k}{\partial x} \ , \ C_s \frac{\partial v_k}{\partial t} = -\frac{\partial i_k}{\partial x}, \\ & \qquad 0 \le x \le 1 \\ & E = v_k(0,t) + R_0 i_k(0,t) \\ & -C \frac{d}{dt} \ v_k(1,t) = -i_k(1,t) + \\ & +g(v_k(1,t)) - \\ & -\frac{1}{R}[v_{k+1} - 2v_k + v_{k-1}](1,t), \\ & \qquad k = 1, \dots, N \end{aligned} \quad (11)$$

We have here N oscillators with tunnel diode and local transmission line - see Brayton and Miranker (1964) - coupled in ring; the "periodicity" of this connection arises from $v_N \equiv v_0$. The delays result from the propagation along the local lossless transmission lines; at the same time the structure of the couplings suggests some discretization of a second derivative. Starting from this remark J.K. Hale introduced a distributed model - see Hale (2004)

$$\begin{aligned} & L_s \frac{\partial i}{\partial t}(x,y,t) = \frac{\partial v}{\partial x}(x,y,t) \ , \\ & C_s \frac{\partial v}{\partial t}(x,y,t) = -\frac{\partial i}{\partial x}(x,y,t) \ , \ 0 \le x \le 1 \\ & E = v(0,y,t) + R_0 i(0,y,t) \\ & -C \frac{\partial v}{\partial t} = -i(1,y,t) + g(v(1,y,t)) - \\ & -\frac{1}{R_1} \ \frac{\partial^2 v}{\partial y^2}(1,y,t) \ , \ y \in S^1 \end{aligned} \quad (12)$$

S^1 being the circle of radius 1. This kind of coupling, diffusive in the sense of the partial differential equations, may be also recognized in other types of networks.

3. DIFFUSIVE COUPLINGS AND CONTROL SIGNALS

Several papers dealing with complex dynamics - Pogromsky (1998); Pogromsky et al (1999); Pogromsky and Nijmeijer (2001) - consider the networked systems from the point of view of the interconnections resulting from control signal synthesis. A sufficiently simple model reads as follows

$$\begin{aligned} \dot{x}_i &= f(x_i) + Bu_i \\ y_i &= Cx_i \ , \ i = 1, \dots, n \end{aligned} \quad (13)$$

where the control signals u_i are defined by the simplest structure of a linear output feedback

$$u_i = -\sum_{j \neq i} \gamma_{ij}(y_i - y_j) \quad (14)$$

This feedback may be viewed as defining a symmetric interconnection matrix with nonnegative entries

$$\gamma_{ij} = \gamma_{ji} \ge 0 \ , \ \sum_j \gamma_{ij} > 0 \quad (15)$$

If a general definition of the synchronization is used - see Blekhman et al (1997) - the signal synthesis of (14) may be done by the minimization of some functional as

$$Q(x_1, \dots, x_n, t) = \sum_{i,j} |x_i(t) - x_j(t)| \quad (16)$$

This is in fact the approach "control-synthesis-feedback" of the networks; the diffusive interconnections (14) may be computed in order to obtain various qualitative behavior of the systems according to synthesis results.

4. SOME CONSIDERATIONS ON OSCILLATORS THAT ARE COUPLED IN RING

We shall consider here the model (11). Its main features are a consequence of the identity of the N oscillators: the same transmission line parameters, the same parameters of the circuit at the two boundaries of the local lines, consequently, the same nonlinear function (the same tunnel diode, the dispersion of the electronic parameters being thus neglected) and a common source $E(t)$.

We shall apply to this system the method of e.g. Răsvan (1975) by introducing first the Riemann invariants, also called forward and backwards waves, for each local oscillator separately

$$v_k(x,t) = u_k^1(x,t) + u_k^2(x,t)$$
$$i_k(x,t) = \sqrt{\frac{C_s}{L_s}}\left(u_k^1(x,t) - u_k^2(x,t)\right) \tag{17}$$

to obtain the transformed system

$$\frac{\partial u_k^1}{\partial t} + \frac{1}{\sqrt{L_s C_s}}\frac{\partial u_k^1}{\partial x} = 0$$
$$\frac{\partial u_k^2}{\partial t} - \frac{1}{\sqrt{L_s C_s}}\frac{\partial u_k^2}{\partial x} = 0$$
$$\left(1 + R_0\sqrt{\frac{C_s}{L_s}}\right)u_k^1(0,t) +$$
$$+\left(1 - R_0\sqrt{\frac{C_s}{L_s}}\right)u_k^2(0,t) = E(t) \tag{18}$$
$$RC\frac{dV_k}{dt} = -Rg(V_k) -$$
$$-R\sqrt{\frac{C_s}{L_s}}\left(u_k^1(x,t) - u_k^2(x,t)\right) +$$
$$+V_{k+1} - 2V_k + V_{k-1}$$
$$u_k^1(1,t) + u_k^2(1,t) = V_k(t)$$

Integrating along the characteristics, as in *(op. cit.)*, we obtain a standard system of coupled delay differential and difference equations as follows

$$T\frac{dw_k}{dt} = -f(w_k) + (1-\rho_0)\eta_k^2(t-\tau) +$$
$$+e(t) + w_{k+1} - 2w_k + w_{k-1}$$
$$\eta_k^1(t) = -\rho_0\eta_k^2(t-\tau) + e(t) \tag{19}$$
$$\eta_k^2(t) = -\eta_k^1(t-\tau) + \delta_0 w_k,$$
$$k = 1,\ldots,N,\ w_0 \equiv w_N$$

where the following notations have been made

$$T = RC\ ,\ \tau = \sqrt{L_s C_s}\ ,$$
$$f(\sigma) = \sqrt{\frac{L_s}{C_s}}\,g(\delta_0\sigma)\ ,\ \delta_0 = R\sqrt{\frac{L_s}{C_s}}$$
$$\rho_o = \frac{1 - R_0\sqrt{\frac{L_s}{C_s}}}{1 + R_0\sqrt{\frac{L_s}{C_s}}}\ ,\ e(t) = \frac{E(t)}{1 + R_0\sqrt{\frac{L_s}{C_s}}}$$

For this system we may formulate several mathematical problems

1^o *Stability* i.e. non-oscillation of each local oscillator under constant input signal e_0.

2^o *Almost linear behavior* of each local oscillator for periodic or almost periodic $e(t)$.

3^o *Synchronization* of the oscillator network i.e. finding conditions for a periodic solution of the overall oscillator network: since there is a unique input periodic signal $e(t)$, all state variables should be periodic with the same period - highest synchronization degree, see Section 1.

4^o *The Turing-Smale problem*: under the constant input signal e_0, with each local oscillator "dead" (i.e. with a globally asymptotically stable equilibrium), find conditions for the interconnections in order to obtain an oscillating ("alive") network.

Worth mentioning that while some of the problems are more or less solved, other ones e.g. the third and the fourth (the Turing Smale problem) are still open for various classes of oscillating systems.

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