BOX-CONSTRAINED STABILIZATION FOR PARAMETRIC UNCERTAIN SYSTEMS

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Abstract: The paper considers a class of continuous-time linear systems with uncertain parameters and develops a methodology for the design of linear feedback laws which, besides stabilization, ensure the constraining of the state-space trajectories inside contractive boxes exponentially decreasing to the equilibrium point. These constraints allow a continuous and individual monitoring of each state variable, despite the parametric uncertainties. The design methodology is based on the positive invariance of the contractive boxes with respect to the dynamics of interval matrix systems. It is shown that the class of matrices that achieve the box-constrained stabilization can be characterized as the solution set of two equivalent linear inequalities. This characterization is further used to derive the design technique as a linear programming problem.

Keywords: constrained stabilization, parametric uncertain systems, interval dynamic systems, matrix inequalities, linear programming.

1. INTRODUCTION

Let us consider a linear system with uncertain parameters described by

$$\dot{x}(t) = \left(A_0 + \sum_{k=1}^{r} \alpha_k A_k \right) x(t) + \left(B_0 + \sum_{l=1}^{q} \beta_l B_l \right) u(t)\,, \quad (1)$$

where A_0, $A_k \in \mathbb{R}^{n \times n}$, B_0, $B_l \in \mathbb{R}^{n \times m}$, are known constant matrices, and α_k, β_l, are independent uncertain parameters that satisfy $|\alpha_k| \leq \hat{\alpha}_k$, $k = 1,\dots,r$, $|\beta_l| \leq \hat{\beta}_l$, $l = 1,\dots q$. Such descriptions are frequently used for coping with the approximate knowledge of the coefficients in the construction of state-space models.

Our objective is the design of a linear feedback $u(t) = Fx(t)$, $F \in \mathbb{R}^{m \times n}$, which stabilizes the parametric uncertain system and constrains its trajectories not to leave a contractive box of the form:

$$X^{\varepsilon}_{de^{ct}} = \{x \in \mathbb{R}^n \mid \| [d_1^{-1} x_1 \cdots d_n^{-1} x_n]^T \|_\infty \leq \varepsilon e^{ct} \}, \quad d_i > 0,\ i = 1,\dots,n,\ c < 0,\ \varepsilon > 0,\ t \geq t_0, \quad (2)$$

once they are initialized inside it. A matrix feedback F fulfilling this objective is called a box-constrained stabilizing matrix. If such an F exists, the system is called *box-constrained stabilizable*.

Intuitively speaking, the box-constrained stabilization means a continuous and individual monitoring of each state variable $x_i(t)$, $i = 1,\dots,n$, not allowing $|x_i(t)| > \varepsilon d_i e^{ct}$ for any $t \geq t_0$, if $|x_i(t_0)| \leq \varepsilon d_i e^{ct_0}$, despite the parametric uncertainties of system (1).
The objective formulated above can be transposed into the mathematical framework of *interval systems* (abbreviated as ISs). Obviously, the description (1) leads to

$$\dot{x}(t) = Ax(t) + Bu(t)\,,\ t \geq 0\,, \quad (3)$$

where the entries of $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and

$B=(b_{ik})\in\mathbb{R}^{n\times m}$ present interval-type uncertainties $a_{ij}^{-}\le a_{ij}\le a_{ij}^{+}$, $b_{ik}^{-}\le b_{ik}\le b_{ik}^{+}$, which, in a compact writing, are componentwise matrix inequalities:

$$A^{-}\le A\le A^{+},\ B^{-}\le B\le B^{+}. \quad (4)$$

The lower and upper bounds $A^{-}=(a_{ij}^{-})$, $A^{+}=(a_{ij}^{+})\in\mathbb{R}^{n\times n}$, $B^{-}=(b_{ik}^{-})$, $B^{+}=(b_{ik}^{+})\in\mathbb{R}^{n\times m}$, can be calculated directly from the known matrices A_0, $A_k\in\mathbb{R}^{n\times n}$, B_0, $B_l\in\mathbb{R}^{n\times m}$ and scalars $\hat{\alpha}_k$, $\hat{\beta}_l$.

The advantage of the IS framework consists in the background available on stability and stabilization, for both continuous- and discrete-time ISs. Therefore the presentation of our research is constructed in terms of ISs, by referring to the box-constrained stability and stabilization of the *open-loop* IS (3)&(4). The closed-loop system

$$\dot{x}(t)=(A+BF)x(t), \quad (5)$$

incorporates the interval uncertainties (4) of both matrices $A\in[A^{-},A^{+}]$, $B\in[B^{-},B^{+}]$ and it will be referred to as the *closed-loop* IS (5)&(4).

The asymptotic stability of IS (3)&(4) with $u(t)\equiv 0$, was one of the most intensively explored properties of ISs, as reflected by the results reported in literature – see the reference list in (Mao and Chu, 2003) and the more recent publications (Chen and Lin, 2004), (Yamaç and Bozkurt, 2004), (Kolev and Petrakieva, 2005), (Zhang *et al.*, 2006). The greater part of these papers provides sufficient conditions, showing that the necessity is valid only for some particular classes of interval matrices. Necessary and sufficient conditions for the stability of arbitrary interval matrices are formulated in (Wang *et al.*, 1994), (Yedavalli, 1999), (Yedavalli, 2001), (Mao and Chu, 2003), (Zhang *et al.*, 2006). Our previous paper (Pastravanu and Voicu, 2004) deserves a special interest since it characterizes the componentwise stability of ISs, and one of its results is further exploitable in the direction of IS stabilization. In a broader context, by regarding interval matrices as matrix polytopes, we should also mention researches on the stability of polytopic systems, such as (Geromel *et al.*, 2006), (Grman *et al.*, 2005), (Liu and Molchanov, 2002), (Kau *et al.*, 2005), (Molchanov and Liu, 2002).

The remark in (Mao and Chu, 2003) that "stabilization of ISs is more difficult than stability analysis" is sustained by the rather scarce literature on the IS stabilization (compared with stability). The stabilization techniques rely on the properties of nonnegative systems (Shafai and Hollot, 1991), quadratic stability and LMIs (Mao and Chu, 2003), (Zhang *et al.*, 2006), generalized antisymmetric stepwise configurations (Hu and Wang, 2000), (Wei, 1994), controllability and spectrum allocation for (A,B) interval pairs (Shashikhin, 2002) and arithmetic intervals (Smagina and Brewer, 2002).

The current work provides a technique for testing the box-constrained stabilizability of IS (3)&(4). Whenever the box-constrained stabilization is possible, the technique also provides a box-constrained stabilizing matrix *F*. The key point of our development consists in transforming a result derived in our previous paper (Pastravanu and Voicu, 2004) into a linear programming (LP) problem. The numerical approach to the LP problem operates as a necessary and sufficient condition in the sense that (i) the absence of a feasible solution means the box-constrained stabilization is not possible, (ii) a feasible solution provides a box-constrained stabilizing matrix. A numerical example illustrates the applicability of the proposed technique.

2. PRELIMINARIES

To address the box-constrained stabilization of IS (3)&(4) we first need a rigorous definition of the box-constrained stability of an IS.

Definition 1. IS (3)&(4) with $u(t)\equiv 0$ is $X_{de^{ct}}$-constrained stable if the boxes $X_{de^{ct}}^{\varepsilon}$, $\varepsilon>0$, introduced by (2) are invariant with respect to the state-space trajectories of (3) with $u(t)\equiv 0$, for any $A\in[A^{-},A^{+}]$. □

Our previous paper (Pastravanu and Voicu, 2004) provides a result that can be used for the characterization of the box-constrained stability of IS (3)&(4).

Theorem 1 Let $\bar{A}=(\bar{a}_{ij})\in\mathbb{R}^{n\times n}$ be an essentially non-negative matrix built from the entries of A^{-}, A^{+} as follows:

$$\begin{aligned}\bar{a}_{ii}&=a_{ii}^{+},\ i=1,\ldots,n,\\ \bar{a}_{ij}&=\sup_{a_{ij}^{-}\le a_{ij}\le a_{ij}^{+}}|a_{ij}|,\ i\ne j,\ i,j=1,\ldots,n.\end{aligned} \quad (6)$$

IS (3)&(4) with $u(t)\equiv 0$ is $X_{de^{ct}}$-constrained stable if and only if the constant $c<0$ and the vector $d>0$ satisfy the inequality:

$$\bar{A}\,d\le cd. \quad (7)$$

Proof: See Corollary 2 in (Pastravanu and Voicu, 2004). □

Inequality (7) provides an easy to apply procedure for the analysis of the $X_{de^{ct}}$- constrained stability of the open-loop IS (3)&(4). However, if we refer to the closed-loop IS (5)&(4), then resolving inequality (7) (i.e. $\overline{A+BF}\,d\le cd$, $A\in[A^{-},A^{+}]$, $B\in[B^{-},B^{+}]$) with respect to matrix $F\in\mathbb{R}^{m\times n}$ is a cumbersome task. Therefore our previous work (Pastravanu and Voicu, 2004) focusing on the analysis of IS componentwise stability could not address the $X_{de^{ct}}$-stabilization of IS (3)&(4) as a direct consequence of the above Theorem. 1

Throughout the paper we use a set of notations for handling matrices and vectors with special structures. These notations have been chosen to support a quick understanding of the contextual message, despite the complexity of the computational approach we intend to develop.

Let $p,q,\pi,\rho \in \mathbb{N}$. $0_{p\times q}$ is the null matrix of size $p\times q$. I_p is the identity matrix of order p.

• For a real matrix $M=(m_{ij})\in \mathbb{R}^{p\times q}$, we introduce the following notations:
$M_{(:,j)}=[m_{1j}\cdots m_{pj}]^T \in \mathbb{R}^p$ is a vector containing the j-th column of M, $j=1,\cdots,q$.
$M|^{\text{vec}}_{pq\times 1}=[M^T_{(:,1)}\dots M^T_{(:,q)}]^T \in \mathbb{R}^{pq}$ is a vector reshaping the elements of matrix M taken columnwise in the ascending order of the column subscript.
If $r=\{r_1,\cdots,r_\eta\}$, $r_1,\dots,r_\eta \in \mathbb{N}$, $1\le r_1<\dots<r_\eta \le p$, is a set of row subscripts for M, then $<M>_r \in \mathbb{R}^{(p-\eta)\times q}$ is the matrix obtained from M by deleting the rows subscripted $r_1,\dots,r_\eta$.
If $M\in \mathbb{R}^{p\times p}$ is a square matrix, then $M^{\text{off}} \in \mathbb{R}^{p\times p}$ preserves the off-diagonal elements of M and has zeros on the main diagonal.

• For a real vector $v=(v_i)\in \mathbb{R}^\rho$, we introduce the following notations:
If $\rho\in\mathbb{N}$ and $\rho=pq$, then

$$v|^{\text{mat}}_{p\times q}=\begin{bmatrix} v_1 & \cdots & v_{(q-1)p+1} \\ v_2 & \cdots & v_{(q-1)p+2} \\ \vdots & \vdots & \vdots \\ v_p & \cdots & v_{qp} \end{bmatrix} \in \mathbb{R}^{p\times q} \quad (8)$$

is a matrix reshaping the elements of the vector v taken in the ascending order of their subscript.
If $r=\{r_1,\dots,r_\eta\}$, $r_1,\dots,r_\eta \in \mathbb{N}$, $1\le r_1<\dots<r_\eta \le p$, is a set of element subscripts for v, then $<v>_r \in \mathbb{R}^{\rho-\eta}$ is the vector obtained from v by deleting the elements subscripted $r_1,\dots,r_\eta$.

• For the construction of a matrix that contains columns selected from two different matrices M^-, $M^+ \in \mathbb{R}^{p\times q}$, we introduce the following notations:
If $M^-_{(:,j)}$, $M^+_{(:,j)}$ are the j-th column of M^- and, respectively, M^+, and $s_j\in\{-1,+1\}$, $j=1,\dots,q$, then $M^{s_j}_{(:,j)}=M^-_{(:,j)}$ for $s_j=-1$ and $M^{s_j}_{(:,j)}=M^+_{(:,j)}$ for $s_j=+1$, i.e. $M^{s_j}_{(:,j)}$ is the j-th column selected in accordance with the value of s_j.
If $s=[s_1\dots s_q]$, $s_j\in\{-1,+1\}$, $j=1,\dots,q$, then the matrix $M^s=[M^{s_1}_{(:,1)}M^{s_2}_{(:,2)}\dots M^{s_q}_{(:,q)}]$ has the columns selected in accordance with the entries of the vector $s\in\{-1,+1\}^q$.

• For two matrices $M\in\mathbb{R}^{p\times q}$, $\mathbf{\Omega}\in\mathbb{R}^{\pi\times\rho}$, the Kronecker product is denoted by $M\otimes\mathbf{\Omega}$ and defined (in a block form) as

$$M\otimes\mathbf{\Omega}=\begin{bmatrix} m_{11}\mathbf{\Omega} & m_{12}\mathbf{\Omega} & \cdots & m_{1q}\mathbf{\Omega} \\ m_{21}\mathbf{\Omega} & m_{22}\mathbf{\Omega} & \cdots & m_{2q}\mathbf{\Omega} \\ \cdots & \cdots & \cdots & \cdots \\ m_{p1}\mathbf{\Omega} & m_{p2}\mathbf{\Omega} & \cdots & m_{pq}\mathbf{\Omega} \end{bmatrix} \in \mathbb{R}^{p\pi\times q\rho}. \quad (9)$$

• For an interval square matrix given as in $(4)_1$, the bar operator $\overline{\bullet}$ defines a constant matrix whose entries are given by (6).

3. CHARACTERIZATON OF BOX-CONSTRAINED STABILIZING MATRICES

In this section we consider the closed-loop IS (5)&(4) and look for a common characterization of all feedback matrices $F=(f_{kj})\in\mathbb{R}^{m\times n}$ that ensure the $X_{de^{ct}}$-constrained stability of IS (5)&(4). We show that all these matrices (regarded as a matrix class) define the solution set of some linear inequalities.

Definition 2. (a) A feedback matrix $F=(f_{kj})\in\mathbb{R}^{m\times n}$ is called an $X_{de^{ct}}$-constrained stabilizing matrix for IS (3)&(4), if the closed loop IS (5)&(4) is $X_{de^{ct}}$-constrained stable (in the sense of Definition 1).
(b) Denote by $\mathcal{F}_{X_{de^{ct}}}$ the set of all $X_{de^{ct}}$-constrained stabilizing matrices for IS (3)&(4). IS (3)&(4) is called $X_{de^{ct}}$-constrained stabilizable if $\mathcal{F}_{X_{de^{ct}}}\neq\varnothing$.□

Theorem 2. Consider the matrix inequalities

$$\forall s\in\{-1,+1\}^m,\ A^+ + B^s F \le G, \quad (10)$$

$$\forall s\in\{-1,+1\}^m,\ -G^{\text{off}} \le (A^- + B^s F)^{\text{off}}, \quad (11)$$

$$Gd \le cd, \quad (12)$$

where $F=(f_{kj})\in\mathbb{R}^{m\times n}$, $G=(g_{ij})\in\mathbb{R}^{n\times n}$. Denote by $\mathcal{F}_{(10)-(12)}=\{F\in\mathbb{R}^{m\times n}\mid \exists G\in\mathbb{R}^{n\times n}:(10)-(12)\text{ true}\}$ the set of all matrices $F\in\mathbb{R}^{m\times n}$ for which there exists $G\in\mathbb{R}^{n\times n}$ such that inequalities (10)-(12) are satisfied. Then $\mathcal{F}_{(10)-(12)}=\mathcal{F}_{X_{de^{ct}}}$.

Proof: Consider an arbitrary $F\in\mathbb{R}^{m\times n}$ defining the state feedback (8), and denote by $\mathbf{\Theta}=(\theta_{ij})$, $i,j=1,\dots,n$, the interval matrix

$$\mathbf{\Theta}=A+BF,\ A\in[A^-,A^+],\ B\in[B^-,B^+], \quad (13)$$

of the closed-loop IS (5)&(4). Let $\theta_{ij}^- \le \theta_{ij}^+$ be the lower and upper bounds of the interval associated with θ_{ij}, $i,j=1,\dots,n$, and consider the matrices $\Theta^- = (\theta_{ij}^-) \in \mathbb{R}^{n\times n}$, $\Theta^+ = (\theta_{ij}^+) \in \mathbb{R}^{n\times n}$. For any θ_{ij}, we can write

$$\begin{aligned}\theta_{ij}^- = a_{ij}^- + \sum_{k=1}^{m} b_{ik}^{-s_{kj}} f_{kj} \le \theta_{ij} = a_{ij} + \sum_{k=1}^{m} b_{ik} f_{ik} \le \\ \le a_{ij}^+ + \sum_{k=1}^{m} b_{ik}^{+s_{kj}} f_{kj} = \theta_{ij}^+,\end{aligned} \tag{14}$$

where, according to the notations in Section2, $b_{ik}^{+s_{kj}} = \begin{cases} b_{ik}^+, & f_{kj} \ge 0 \\ b_{ik}^-, & f_{kj} < 0 \end{cases}$ and $b_{ik}^{-s_{kj}} = \begin{cases} b_{ik}^-, & f_{kj} \ge 0 \\ b_{ik}^+, & f_{kj} < 0 \end{cases}$.

The expressions of θ_{ij}^-, θ_{ij}^+ given by (14) show that Θ^+ and Θ^- can be described columnwise by

$$\begin{aligned}\Theta_{(:,j)}^- &= A_{(:,j)}^- + \sum_{k=1}^{m} B_{(:,k)}^{-s_{kj}} f_{kj}, \\ \Theta_{(:,j)}^+ &= A_{(:,j)}^+ + \sum_{k=1}^{m} B_{(:,k)}^{+s_{kj}} f_{kj},\ j=1,\dots,n.\end{aligned} \tag{15}$$

Let $F \in \mathcal{F}_{(10)-(12)}$. Since inequalities (10), (11) involve all matrices B^s built columnwise for $s \in \{-1,+1\}^m$, the fulfillment of (10) and (11) ensures $\Theta_{(:,j)}^+ \le G_{(:,j)}$ and $<-G_{(:,j)}>_{\{j\}} \le <\Theta_{(:,j)}^->_{\{j\}}$, for all $j=1,\dots,n$. This is equivalent to $\theta_{ij}^+ \le g_{ij}$, $i,j=1,\dots,n$, and $-g_{ij} \le \theta_{ij}^-$, $i \ne j$, $i,j=1,\dots,n$, respectively. By using the $\bar{\theta}_{ij}$ notation, we get $\bar{\theta}_{ij} \le g_{ij}$, $i,j=1,\dots,n$. In a compact writing, we have $\bar{\Theta} \le G$, which, together with inequality (12), imply $\bar{\Theta} d \le cd$. Theorem 1 guarantees the $X_{de^{ct}}$-constrained stability of the closed-loop IS (5)&(4), i.e. $F \in \mathcal{F}_{X_{de^{ct}}}$. Thus, we have proven that $\mathcal{F}_{(10)-(12)} \subseteq \mathcal{F}_{X_{de^{ct}}}$.

For the counterpart, let us consider $F \in \mathcal{F}_{X_{de^{ct}}}$, i.e. the closed-loop IS (5)&(4) is $X_{de^{ct}}$- constrained stable. Theorem 1 ensures the fulfillment of inequality $\bar{\Theta} d \le cd$.

On the other hand, the expressions (14) of θ_{ij}^-, θ_{ij}^+ show that $\theta_{ij}^- \le a_{ij}^- + \sum_{k=1}^{m} b_{ik}^{\sigma_{ik}} f_{kj}$ and $a_{ij}^+ + \sum_{k=1}^{m} b_{ik}^{\sigma_{ik}} f_{kj} \le \theta_{ij}^+$ for any choice of $\sigma_{ik} \in \{-1,+1\}$, which includes the case of a common choice per column, i.e. for all $i=1,\dots,n$, $\sigma_{ik} = s_k \in \{-1,+1\}$. This means

$$\forall s \in \{-1,+1\}^m:\ A^+ + B^s F \le \Theta^+,\ \Theta^- \le A^- + B^s F. \tag{16}$$

Since $\Theta^+ \le \bar{\Theta}$ and $(-\bar{\Theta})^{\text{off}} \le (\Theta^-)^{\text{off}}$, inequalities (10)–(12) are true for the considered F and $G = \bar{\Theta}$, i.e. $F \in \mathcal{F}_{X_{de^{ct}}}$, proving that $\mathcal{F}_{X_{de^{ct}}} \subseteq \mathcal{F}_{(10)-(12)}$. □

Remark 1. Theorem 2 shows that the problem of $X_{de^{ct}}$-constrained stabilization for IS (3)&(4) does not require the exploration of all 2^{mn} vertices of the polytope defined by the interval matrix $[B^-, B^+]$. The proof of Theorem 2 reveals that from 2^{mn} possible tests, only 2^m tests are meaningful. In other words, by checking those 2^m vertices specified by the theorem, one gets complete information about the extreme values of the interval entries of the closed-loop matrix $A+BF$. □

Although inequalities (10)-(12) are linear, their matrix form is still inconvenient for the automatic manipulation in a scientific software environment. Therefore we reorganize the matrix inequalities (10)-(12) in the standard form of a linear inequality with appropriate dimensions $M\omega \le v$, $M \in \mathbb{R}^{p\times q}$, $\omega \in \mathbb{R}^q$, $v \in \mathbb{R}^p$, where the vector ω collects the elements of the matrices F and G.

Theorem 3. Consider the inequality

$$\left[\begin{array}{cc} I_n \otimes B^{[+1\cdots+1]} & -I_{n^2} \\ \vdots & \vdots \\ I_n \otimes B^{[-1\cdots-1]} & -I_{n^2} \\ \hline -<I_n \otimes B^{[+1\cdots+1]}>_r & -<I_{n^2}>_r \\ \vdots & \vdots \\ -<I_n \otimes B^{[-1\cdots-1]}>_r & -<I_{n^2}>_r \\ \hline 0_{n\times mn} & d^T \otimes I_n \end{array}\right] \begin{bmatrix} \varphi \\ \gamma \end{bmatrix} \le \left[\begin{array}{c} -A^+|_{n^2\times 1}^{\text{vec}} \\ \vdots \\ -A^+|_{n^2\times 1}^{\text{vec}} \\ \hline <A^-|_{n^2\times 1}^{\text{vec}}>_r \\ \vdots \\ <A^-|_{n^2\times 1}^{\text{vec}}>_r \\ \hline cd \end{array}\right] \tag{17}$$

where $r = \{r_1,\dots,r_n\}$, $r_j = (j-1)n + j$, $j=1,\dots,n$, and $\varphi \in \mathbb{R}^{nm}$, $\gamma \in \mathbb{R}^{n^2}$. Denote by $\Phi_{(17)}$ the set of all vectors $\varphi \in \mathbb{R}^{nm}$ for which there exists $\gamma \in \mathbb{R}^{n^2}$ such that inequality (17) is satisfied, i.e $\Phi_{(17)} = \{\varphi \in \mathbb{R}^{mn} \mid \exists \gamma \in \mathbb{R}^{n^2} : (17) \text{ true}\}$.

Let $\mathcal{F}_{(17)} = \{F \in \mathbb{R}^{m\times n} \mid F = \varphi|_{m\times n}^{\text{mat}}, \varphi \in \Phi_{(17)}\}$ be the set of matrices obtained by reshaping the vectors from $\Phi_{(17)}$.

Then $\mathcal{F}_{(17)} = \mathcal{F}_{X_{de^{ct}}}$.

Proof: We prove that inequality (17) is equivalent to inequalities (10)-(12).

The first $2^m n^2$ scalar inequalities in (17) are equivalent to the matrix inequalities (10) written for all 2^m combinations of signs in the vector s, i.e. from $s = [+1\dots+1]$ to $s = [-1\dots-1]$. Indeed, for each $s \in \{-1,1\}^m$, in (17) we identify a group of n^2 inequalities of the form $(I_n \otimes B^s)\varphi - \gamma \le -A^+|_{n^2\times 1}^{\text{vec}}$. By considering the matrices $F = \varphi|_{m\times n}^{\text{mat}}$, $G = \gamma|_{n\times n}^{\text{mat}}$, this inequality is equivalent to $B^s F_{(:,j)} - G_{(:,j)} \le -A^+_{(:,j)}$, $j=1,\ldots,n$, and, finally, to (10).

An analogous construction shows that the next $2^m(n^2-n)$ scalar inequalities in (17) are equivalent to the matrix inequalities (11) written for all 2^m combinations of signs in the vector s. The only difference consists in deleting the rows $r_j=(j-1)n+j$, $j=1,\ldots,n$, from each group of n^2 inequalities of the form $-\left(I_n \otimes B^s\right)\varphi - \gamma \le A^-|^{\text{vec}}_{n^2\times 1}$, for $s \in \{-1,1\}^m$, which means the columnwise writing $-<B^s>_{\{j\}}F_{(:,j)} - <G_{(:,j)}>_{\{j\}} \le <A^-_{(:,j)}>_{\{j\}}$, $j=1,\ldots,n$, or equivalently (11).

Finally, the last n scalar inequalities in (17) are equivalent to (12).

Consequently, $\mathcal{F}_{(17)} = \mathcal{F}_{(10)-(12)}$ and, taking Theorem 2 into account, we have $\mathcal{F}_{(17)} = \mathcal{F}_{X_{de^{ct}}}$. □

4. NUMERICAL TRACTABILITY

Once we know that $\mathcal{F}_{X_{de^{ct}}}$ represents the solution set of the linear inequality (17), we are interested in developing a computational procedure for finding a concrete $X_{de^{ct}}$ - constrained stabilizing feedback matrix $F \in \mathcal{F}_{X_{de^{ct}}}$, when $\mathcal{F}_{X_{de^{ct}}} \ne \varnothing$. We propose a linear programming (LP) approach since the linear inequality (17) can be exploited for defining the constraints, and the minimization can refer to the decreasing rate of the invariant sets (4) which ensure the constrained stability of the closed loop IS.

Theorem 4. Consider the LP problem that minimizes the objective function:

$$J(\varphi,\gamma,\lambda)=\lambda \tag{18}$$

with the constraints

$$\left[\begin{array}{ccc} I_n\otimes B^{[+1\cdots+1]} & -I_{n^2} & 0_{n^2\times 1} \\ \vdots & \vdots & \vdots \\ I_n\otimes B^{[-1\cdots-1]} & -I_{n^2} & 0_{n^2\times 1} \\ \hdashline -<I_n\otimes B^{[+1\cdots+1]}>_r & -<I_{n^2}>_r & 0_{(n^2-n)\times 1} \\ \vdots & \vdots & \vdots \\ -<I_n\otimes B^{[-1\cdots-1]}>_r & -<I_{n^2}>_r & 0_{(n^2-n)\times 1} \\ \hdashline 0_{n\times mn} & d^T\otimes I_n & -d \\ 0_{1\times mn} & 0_{1\times n^2} & 1 \end{array}\right] \begin{bmatrix}\varphi\\ \gamma\\ \lambda\end{bmatrix} \le \left[\begin{array}{c} -A^+|^{\text{vec}}_{n^2\times 1} \\ \vdots \\ -A^+|^{\text{vec}}_{n^2\times 1} \\ \hdashline <A^-|^{\text{vec}}_{n^2\times 1}>_r \\ \vdots \\ <A^-|^{\text{vec}}_{n^2\times 1}>_r \\ \hdashline 0_{n\times 1} \\ c \end{array}\right] \tag{19}$$

where $r=\{r_1,\ldots,r_n\}$, $r_j=(j-1)n+j$, $j=1,\ldots,n$, and $\varphi\in\mathbb{R}^{nm}$, $\gamma\in\mathbb{R}^{n^2}$, $\lambda\in\mathbb{R}$. Denote by Φ_{LP} the set of all vectors $\varphi\in\mathbb{R}^{nm}$ which are solutions to LP. Define $\mathcal{F}_{LP}=\{F\in\mathbb{R}^{m\times n} \mid F=\varphi|^{\text{mat}}_{mn\times 1}, \varphi\in\Phi_{LP}\}$.

Then

(a) $\mathcal{F}_{LP} \subseteq \mathcal{F}_{X_{de^{ct}}}$.

(b) $\mathcal{F}_{LP} \equiv \varnothing \Rightarrow \mathcal{F}_{X_{de^{ct}}} \equiv \varnothing$.

Proof: a) Let $F\in\mathcal{F}_{LP}$ and consider the corresponding solution $\varphi\in\mathbb{R}^{nm}$, $\gamma\in\mathbb{R}^{n^2}$, $\lambda\in\mathbb{R}$ of the LP problem. This means λ fulfills the constraint $\lambda\le c$, and $\lambda d\le cd$. Consequently, $\varphi\in\mathbb{R}^{nm}$, $\gamma\in\mathbb{R}^{n^2}$, satisfy the inequality (17) i.e. $F\in\mathcal{F}_{(17)}$, and Theorems 3 guarantees $F\in\mathcal{F}_{X_{de^{ct}}}$.

b) Assume that $\mathcal{F}_{LP}\equiv\varnothing$, but $\mathcal{F}_{X_{de^{ct}}}\ne\varnothing$. According to Theorem 3, inequality (17) has solution(s) $\varphi\in\mathbb{R}^{nm}$, $\gamma\in\mathbb{R}^{n^2}$, and these vectors together with $\lambda=c$ also satisfy the constraints (19) of the LP problem. Hence, the LP problem is feasible and $\mathcal{F}_{LP}\ne\varnothing$, fact which, by contradicting the hypothesis, completes the proof. □

Remark 2. Theorem 4 provides a computable necessary and sufficient condition for the $X_{de^{ct}}$ - constrained stabilizability of IS (3)&(4). The robust numerical tractability of the LP problems ensures the practical applicability of the result. The LP solver returns an unfeasible solution if and only if IS (3)&(4) is not $X_{de^{ct}}$ - constrained stabilizable; otherwise any feasible solution can be used as a $X_{de^{ct}}$ - constrained stabilizing feedback matrix. □

5. ILLUSTRATIVE EXAMPLE

To illustrate our approach, we consider the linear system with uncertain parameters described by (1) for $r=4$ and $q=2$ with the following numerical values of the matrices:

$$A_0=\begin{bmatrix}2.25 & -4.875\\ -2.625 & 2.20\end{bmatrix},\ A_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ A_2=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},$$

$$A_3=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, A_4=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, B_0=\begin{bmatrix}-1.875\\ 1.10\end{bmatrix}, B_1=\begin{bmatrix}1\\ 0\end{bmatrix}, B_2=\begin{bmatrix}0\\ 1\end{bmatrix}$$

for $|\alpha_1|\le 0.25$, $|\alpha_2|\le 0.125$, $|\alpha_3|\le 0.125$, $|\alpha_4|\le 0.2$, $|\beta_1|\le 0.125$, $|\beta_2|\le 0.1$. Obviously, this description can be equivalently written as the IS (3)&(4) for the interval matrices defined by

$$A^-=\begin{bmatrix}2 & -5\\ -2.75 & 2\end{bmatrix},\ A^+=\begin{bmatrix}2.50 & -4.75\\ -2.50 & 2.40\end{bmatrix},\ B^-=\begin{bmatrix}-2\\ 1\end{bmatrix}^T,$$

$$B^+=\begin{bmatrix}-1.75\\ 1.20\end{bmatrix}^T.$$

Notice that IS (3)&(4) with $u(t)\equiv 0$ is not asymptotically stable since there exist matrices $A\in[A^-,A^+]$ that are not Hurwitz stable (for example, A^- and A^+). Obviously, the considered system cannot be $X_{de^{ct}}$-constrained stable for arbitrary constant $c<0$ and positive vector $d>0$.

For $d=[2\ \ 1]^T$ and $c=-0.1$ we want to find a feedback matrix F that allows the $X_{de^{ct}}$-constrained stabilization of the considered IS and we use Theorem 4. The LP problem is solved by using the **linprog** function from the Optimization Toolbox for MATLAB. The numerical solution is

$\varphi \in \mathbb{R}^2$, $\varphi^T = [2.3864 \;\; -3.5892]^T$,

$\gamma \in \mathbb{R}^4$, $\gamma^T = [-1.6761,\, 0.3636,\, 2.4284,\, -1.1892]^T$

and $\lambda = -0.4619$. We conclude that the feedback matrix $F = \varphi|^{\text{mat}}_{1\times 2} = [2.3864 \;\; -3.5892]$ guarantees that the closed-loop IS (5)&(4) is not only $X_{de^{ct}}$ - constrained stable, but also $X_{de^{\lambda t}}$ - constrained stable, with $\lambda = -0.4619 < c = -0.1$. In terms of positive invariance, the sets $X^{\varepsilon}_{de^{\lambda t}}$ decrease faster than $X^{\varepsilon}_{de^{ct}}$. For the above F, the matrix $\bar{\Theta}$ of the closed-loop IS (5)&(4) is given by the elements of vector γ reshaped as

$$\bar{\Theta} = G = \gamma|^{\text{mat}}_{2\times 2} = \begin{bmatrix} -1.6761 & 2.4284 \\ 0.3636 & -1.1892 \end{bmatrix}.$$

6. CONCLUSIONS

In comparison with the standard concept of stabilization, the box-constrained-stabilization ensures supplementary properties to the closed-loop trajectories of a parametric uncertain system. These supplementary properties are offered by the exponentially decreasing boxes (2), which are invariant with respect to the closed-loop trajectories. The proposed design technique is numerically robust and operates in a single step, either providing a feedback matrix that achieves the box-constrained stabilization, or deciding that the considered system is not box-constrained stabilizable.

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