STUDY OF KALMAN FILTERING – AN EXAMPLE RELATED TO UKF

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Abstract: An alternative approach to Extended Kalman Filter (EKF) has emerged over the last few years, namely the unscented Kalman filter (UKF). This filter claims both higher accuracy and robustness for nonlinear models. This paper investigates the accuracy for nonlinear measurement models in particular by comparing the performance of EKF and UKF for two tracking models having nonlinear measurements.

Keywords: Kalman filtering, EKF, UKF

1. INTRODUCTION

The problem of *state estimation* concerns the task of estimating the state of a process while only having access to noisy and/or inaccurate measurements from that process. It is a very ubiquitous problem setting, encountered in almost every discipline within science and engineering. The most commonly used type of state estimator is the Kalman filter. It is an optimal estimator for linear systems, but unfortunately very few systems in real world are linear. This linearization does however pose some problems, e.g. it can result in nonstable estimates (Julier, 1996).

The development of better estimator algorithms for nonlinear systems has therefore attracted a great deal of interest in the scientific community, because improvements here will undoubtedly have great impact in a wide range of engineering fields.

A state space model is a mathematical model of a process, where the process' *state* **x** is represented by a numerical vector. State-space models actually consists of two separate models: the *process model*, which describes how the state propagates in time based on external influences, such as input and noise; and the *measurement model*, which describe how measurements **z** are taken from the process, typically simulating noisy and/or inaccurate measurements.

2. KALMAN FILTERING

2.1 Extended Kalman Filter - EKF.

Kalman filter deals with the general problem to trying to estimate the state $x \in \Re^n$ of a discrete time controlled process that is governed by a *linear* stochastic difference equation. The question is what will happen if the process subject to estimation and/or relation between measurements and process are nonlinear. A Kalman filter that linearize about the current mean and covariance is referred in literature as an *extended Kalman filter* or EKF. In practice EKF often exhibits instability, particularly when targets cross and it can be extremely sensitive to initial state and error covariance values as well as the selected process noise covariance. The EKF is based on first order Taylor series expansion (linearization) about the state estimate but the accuracy of this expansion in series breaks down if the estimated state is too far from the true state. EKF seems to be well-suited to handle gentle nonlinearities.

It is important to specify that an EKF basically drawback is that the distributions (or the densities in the continuous case) belong the various random variables are not normal anymore after they undergo those nonlinear transformations. EKF is just an *ad-hoc* state estimator that just approximate the Bayes' rule optimality through linearization. There are interesting papers in the literature which develop

EKF variations using methods that preserve the normal distributions passing through the nonlinear transformations.

2.2 *Unscented Kalman Filter – UKF.*

Let's point out shortly the rough approximations of the EKF: EKF uses the first order terms from Taylor expansion; there are big errors when the models have strong nonlinearities; local linearity hypothesis is no more valid when higher order terms became significant. To eliminate these deficiencies it was looked for another approximation. The unscented transformation (UT) is a method for calculating the statistics of a random variable which undergoes a nonlinear transformation and builds on the principle that is easier to approximate a probability distribution than an arbitrary nonlinear function.

The problem of UT transformation is to propagate $\boldsymbol{x}$ – an $\boldsymbol{n_x}$ dimensional random variable – through a nonlinear function $g : R^{n_x} \mapsto R^{n_y}$ to generate $\boldsymbol{y}$: $y = g(x)$. The problem of propagating Gaussian random variables through a nonlinear function can also be approached using another technique, namely the *unscented transform*. Instead of linearizing the functions, this transform uses a set of points, and propagates them through the actual nonlinear function, eliminating linearization altogether. The points are chosed such that their mean, covariance, and possibly also higher order moments, match the Gaussian random variable. Mean and covariance can be recalculated from the propagated points, yielding more accurate results compared to ordinary function linearization. The underlying idea is also to approximate the probability distribution instead of the function. This strategy typically does both decrease the computational complexity, while at the same time increasing estimate accuracy, yielding faster, more accurate results.

The underlying method of unscented transform was first proposed by Uhlmann; he laid out the framework for representing a Gaussian random variable in N dimensions using $2N+1$ samples, called *sigma points*. He utilized the matrix square root and covariance definitions to selectthese points in such a way that he had the same covariance as the Gaussian they approximated. Skewness was avoided by selecting the points in a symmetric way, such that any approximation error would only originate from the fourth and higher moments. Usage of the unscented transform in Kalman filtering was then presented by Julier; he introduced the *Unscented Kalman filter* (UKF), which approximates the state estimate using sigma points.

A limitation associated with the unscented Kalman filter is that it has a lower bound on the *safe spread* of the sigma points, meaning the distance between the points in state space. Sigma point spreads below this bond are not guaranteed to yield positive semidefinite correlation matrices. This distance also increases with the dimension of the state space, a limitation that may cause problems in highly nonlinear models, since high sigma point spread may result in sampling of non-local features.

The technique presented here is therefore based on the *scaled unscented transform*, which provides an additional tuning parmeter,α, compared to the original unscented transform. This parameter is used to arbitrary control the spread of the sigma-points, while at the same time guaranteeing positive semidefinite covariance matrices. Even models of high dimensonality can then keep a tight sigma point spread to avoid nonlocal effects.

3. COMPARISON EKF-UKF

The UKF compares favorably to the EKF in two other aspects as well. The UKF, like the EKF, forces the posterior density to be Gaussian, but the posterior mean and covariance are accurate to a third-order Taylor series expansion compared to first-order accuracy for the EKF. Finally, the UKF has the same order of computational complexity as the EKF. With these credentials, the UKF was expected to consistently outperform the EKF.

The experiments described here aims at determining whether there are any difference between EKF and UKF for practical tracking applications, having linear process models and nonlinear measurement models.

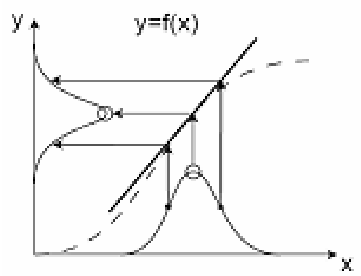

Fig.1. Illustration of how the Extended Kalman filter linearizes a nonlinear function around the mean of a Gaussian distribution, and thereafter propagates the mean and covariance through this linearized model

Process Model. The basis for the experiment is an aeroplane, modelled linearly for position and velocity respectively, driven by white noise acceleration. The model

$$\dot{p}_x = v_x,\ \dot{p}_y = v_y,\ \dot{v}_x = a_x,\ \dot{v}_y = a_y$$

yields the following discrete time process model when assuming zero order hold with timestep $Ts = 1$.

$$\begin{bmatrix} p_x \\ p_y \\ v_x \\ v_y \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ v_x \\ v_y \end{bmatrix}_k + \begin{bmatrix} 0 \\ 0 \\ a_x \\ a_y \end{bmatrix}_k$$

where the accelerations ax and ay are modelled as uncorrelated white noise with a variance of 0.5.

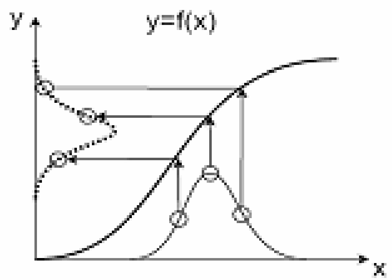

Fig.2. Illustration of how the unscented Kalman filter propagates sigma-points from a Gaussian distribution through a nonlinear function, and recreates a Gaussian distribution, by calculating the mean and covariance of the results

The process is assumed to start in the following state: $\mathbf{x}_0 = [-200\ 200\ 4\ 0]^T$ with the squared error metric $(\mathbf{x}-\hat{\mathbf{x}})^T(\mathbf{x}-\hat{\mathbf{x}})$ for estimation accuracy.

Tracking by Radar. Radar tracking can be modelled with a measurement model observing distance and angle to the target:

$$\begin{bmatrix} d \\ \Theta \end{bmatrix} = \begin{bmatrix} \sqrt{p_x^2 + p_y^2} \\ arctg(p_y / p_x) \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

where the measurement model is clearly nonlinear. The radar is assumed to be positioned in the coordinates (0,0) with measurement noises $n1$ and $n1$, having a variance of 200 and 0.003, respectively.

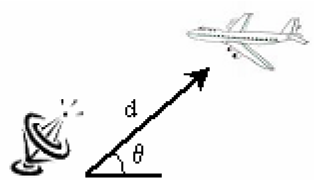

Fig.3. Tracking of plane motion by means of a radar

Tracking by Triangulation. Tracking by triangulation can similarly be modelled with a measurement model observing distances to the target from two observators:

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \sqrt{(p_x - p_{1x})^2 + (p_y - p_{1y})^2} \\ \sqrt{(p_x - p_{2x})^2 + (p_y - p_{2y})^2} \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

where the measurement model is also clearly nonlinear. The observators are assumed to be positioned in the coordinates (−300,0) and (300,0), with measurement noises $n1$ and $n2$, both having a variance of 200.

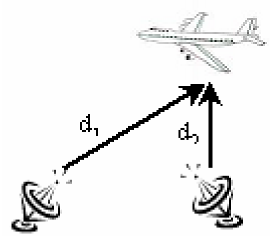

Fig.3. Tracking of plane motion by means of triangulation

Results. A simulation/estimation experiment was run 10000 times. Each time, a simulated plane trajectory were estimated over 80 time steps, by both EKF and UKF for both of the observation models. The estimation accuracy results are shown in the table below, with accuracy distribution plots in the accompanying figure. The MSE estimate variance is calculated from the empirical error distribution, using formula: $VAR(MSE) = VAR(\text{eroare}) / N$.

Table 1. Comparison EKF-UKF by MSE (Accuracy variance is given in parenthesis)

Model	MSE la EKF	MSE la UKF
Radar	174,4 (5,00)	116,9 (0,363)
Triangulation	185,2 (3,15)	183,1 (2,81)

The radar model, having measurements involving the highly nonlinear arcus-tangent, shows a wider difference in the estimation accuracy between EKF and UKF, compared to the triangulation model which only has Pythagoras' measurements, being significantly more linear. It can further be seen that UKF seems to show a higher degree of robustness, having fewer estimates with errors above 1000 for both of the models. Graphical representation of the estimation error distribution proves that the two estimators produce similar results for both models (figure 5). Figure 6 illustrate UKF action mode with an arbitrary trajectory of a target.

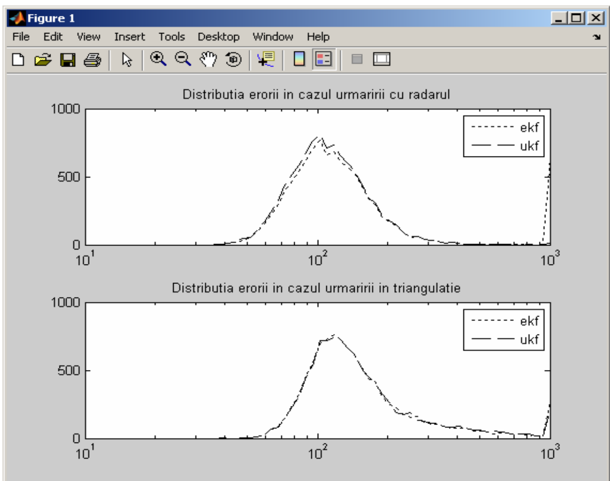

Fig.5. EKF and UKF estimation errors in cases of radar and triangulation tracking

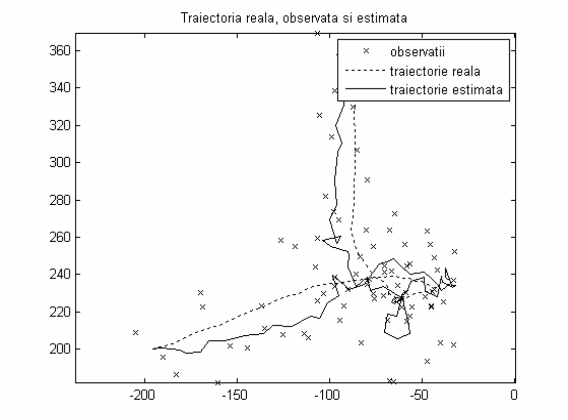

Fig.6. Example illustration of observations, true trajectory and estimated trajectory for the experiments using the UKF

4. CONCLUSION

This paper did therefore compare the relative estimation accuracy of UKF compared to EKF for linear state space models with nonlinear measurements. The relative advantage of using UKF does seem to increase with the degree of nonlinearity in the measurement model.

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