KALMAN -TYPE ESTIMATION FOR SYSTEMS CORRUPTED WITH ADDITIVE AND MULTIPLICATIVE WHITE NOISE

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Abstract: The purpose of the paper is to present an estimation method of the state vector of linear stochastic systems corrupted both with additive and multiplicative white noise. The observer design is based on an H_2-type optimization problem for the considered class of stochastic systems. The solution is given in terms of the solution of a specific system of Riccati and Lyapunov equations which in the absence of the multiplicative noise, reduces at the well-known filtering equation in Kalman-Bucy filtering. A numerical example illustrates the theoretical developments.

Keywords: Stochastic systems, Multiplicative noise, Stability, Estimation theory, Kalman filters, Riccati equations, Iterative methods.

1. INTRODUCTION

The filtering problem received a major attention from the early formulation and developments due to E. Hopf and N. Wiener in the 1940's in the military context and continuing with the famous results of R.E. Kalman and R.S. Bucy derived two decades later (see (Kalman, 1960) and (Kalman and Bucy, 1961)). Since that period, the filters have been widely used in many areas of engineering sciences including aerospace applications, image processing, geophysics, etc. A comprehensive survey on linear filtering and estimation can be found in (Kailath, 1974). An important property of a filter is its robustness with respect to the modelling uncertainty of the system whose state is estimated. It is a known fact that the filter performance significantly degradates in the presence of modelling uncertainty. Many papers have been devoted to the robust filtering in the presence of parametric uncertainty (see *e.g.* (Kalman, 1963), (Mangoubi, 1998)). However, there are applications in which the system parameters are subjected to random perturbations, requiring thus a stochastic modelling of the uncertain dynamic system. A natural way to introduce stochastic uncertainty is referred as *multiplicative (or state-dependent) noise*. Such stochastic systems have been intensively studied over the last four decades (see *e.g.* (Aström, 1970), (Wonham, 1970)) and a characteristic feature of them is that the random perturbations cease when the system reaches its equilibrium state. The stochastic systems considered in this paper are also corrupted with additive and multiplicative white noises. By contrast with the developments in (Stoica and Tiba, 2007) where the multiplicative white noise is present only in the state equation, in this paper the output equation also includes a state-dependent noise component enlarging thus the area of applications.

A deterministic Luenberger observer-based structure is adopted because of its simple implementation. Of course, some better theoretical results could be obtained if a stochastic filter with multiplicative white noise would be considered but in this case difficult implementation occur since the component with multiplicative noise (also called *diffusion term*) is not measured. In the particular case when the multiplicative noise is missing the problem simply reduces to the classical Kalman filtering.

The paper is organized as follows: the next section includes some notations, definitions and known results useful for the following developments. In the final part of Section 2, the statement of the filtering problem is given. The main result is presented and proved in the third section. It is illustrated by a numerical case study given in Section 4.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Definitions and preliminary results

Consider the following stochastic system corrupted with multiplicative and additive white noise, described by the Itô equations:

$$\begin{aligned} dx(t) &= Ax(t)dt + Dx(t)d\xi(t) + Bd\beta(t) \\ dy(t) &= Cx(t)dt + Gx(t)d\nu(t) + d\eta(t) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the system state vector, $y \in \mathbb{R}^p$ is the measurement, ξ, β, ν and η are zero-mean independent Wiener processes on a given probability field $(\Omega, \mathcal{F}, \mathcal{P})$. The matrices $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $G \in \mathbb{R}^{p\times n}$ are assumed given.

The next two definitions will be used in the following developments.

Definition 1. The stochastic system with multiplicative white noise

$$dx(t) = Ax(t)dt + Dx(t)d\xi(t) \tag{2}$$

is called *exponentially stable in mean square (ESMS)* if there exist $\beta \geq 1$ and $\alpha > 0$ such that

$$E\left[\left|\Phi(t,t_0)\right|^2\right] \leq \beta e^{-\alpha(t-t_0)}$$

where $E[\cdot]$ denotes the expectation and $\Phi(t,t_0)$ is the fundamental matrix solution of system (2), namely if $x(t,t_0,x_0)$ is the solution of (2) with the initial condition x_0 at t_0 then $x(t,t_0,x_0) = \Phi(t,t_0)x_0$, a.s. The norm $|\Phi|$ stands for the square root of the largest eigenvalue of $\Phi^T\Phi$.

For a symmetric matrix P, throughout the paper $P \geq 0 (P \leq 0)$ means that P is positive (negative) semidefinite and $P_1 \geq P_2$ means that $P_1 - P_2 \geq 0$.

The stochastic stability of a system with state-dependent noise of form (1) is characterized by the following result whose proof may be found for instance in (Hasminskii, 1980).

Proposition 1. The system (1) is ESMS if and only if there exists a symmetric matrix $X > 0$ such that

$$A^T X + XA + D^T XD < 0 .$$

The next result extends the well-known definition of the H_2 norm defined in the deterministic framework, at stochastic systems with state-dependent noise (see *e.g.* (Dragan *et al.*, 1992)).

Proposition 2. Assume that G denotes an ESMS stochastic system with multiplicative and additive white noise described by the state equation

$$dx(t) = Ax(t)dt + Dx(t)d\xi(t) + Bd\beta(t) \tag{3}$$

with the quality output

$$z(t) = Cx(t) . \tag{4}$$

Then $\lim_{t\to\infty} E\left[\left|z(t)\right|^2\right]$ with $z(t)$ determined for null initial conditions of (3) is well defined and by definition, the H_2-type norm of G is

$$\|G\|_2 := \left\{\lim_{t\to\infty} E\left[\left|z(t)\right|^2\right]\right\}^{\frac{1}{2}} .$$

Moreover, the H_2-type norm of the system (3), (4), has the expression $\|G\|_2 = Tr\left(C^T P_c C\right)$ where $Tr(\cdot)$ denotes the trace of the matrix $(\cdot)$ and P_c is the *controllability Gramian* defined as the positive semidefinite solution of the Lyapunov-type equation

$$AP_c + P_c A^T + DP_c D^T + BB^T = 0 . \tag{5}$$

The next preliminary result presented in this subsection directly follows from the monotonicity properties of the stabilizing solutions of algebraic Riccati equations (see *e.g.* (Wimmer, 1985)).

Proposition 3. If $X \geq 0$ and $\tilde{X} \geq 0$ are the stabilizing solutions of the Riccati equations

$$AX + XA^T - XC^T CX + P_1 = 0$$

and

$$A\tilde{X} + \tilde{X}A^T - \tilde{X}C^T C\tilde{X} + P_2 = 0 ,$$

respectively (that is $A - XC^T C$ and $A - \tilde{X}C^T C$ respectively, are Hurwitz), where $P_2 \geq P_1$, then $\tilde{X} \geq X$.

Finally, the following known result will be used in the proof of the main result in the next section.

Proposition 4. If X_1 and X_2 are symmetric and $X_1 \geq X_2 \geq 0$ then $Tr(X_1) \geq Tr(X_2) \geq 0$.

2.2 Problem formulation

Given the stochastic ESMS system with state-dependent and additive white noise (1), determine the Luenberger observer-type filter of form

$$d\hat{x}(t) = A\hat{x}(t)dt + L\left(dy(t) - C\hat{x}(t)dt\right) \qquad (6)$$

such that $A - LC$ is Hurwitz and H_2 norm of the mapping

$$\begin{bmatrix} \beta(t) \\ \eta(t) \end{bmatrix} \to e(t) := x(t) - \hat{x}(t)$$

is minimized.

Remark 1. The observer (6) has the structure of a deterministic system. A more complex structure including state-dependent noises components in (6) may provide better estimation results but in this case implementation problems occur since these noises cannot be directly measured.

3. MAIN RESULT

The solution of the estimation problem formulated above is given by the following result:

Theorem 1. The optimal filter gain L for which $A - LC$ is Hurwitz and $\lim_{t\to\infty} E\left[\left|x(t) - \hat{x}(t)\right|^2\right]$ is minimized is given by

$$L = XC^T K^{-1} \qquad (7)$$

where X is the stabilizing solution of the Riccati equation

$$AX + XA^T - XC^T K^{-1}CX + BB^T + DYD^T = 0\,, \qquad (8)$$

Y stands for the solution of the equation

$$AY + YA^T + DYD^T + BB^T = 0 \qquad (9)$$

and

$$K := I + GYG^T\,. \qquad (10)$$

Proof. Coupling the systems (1) and (6) one obtains

$$dx = Axdt + Dxd\xi + Bd\beta$$
$$d\hat{x} = (A - LC)\hat{x}dt + LCxdt + LGxd\nu + Ld\eta$$

Subtracting the above equations it results the following equivalent system

$$de = (A - LC)edt + Dxd\xi - LGxd\nu + Bd\beta - Ld\eta$$
$$dx = Axdt + Dxd\xi + Bd\beta,$$

namely,

$$\begin{aligned}\begin{bmatrix} de \\ dx \end{bmatrix} &= \begin{bmatrix} A-LC & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} d\xi \\ &\quad + \begin{bmatrix} 0 & -LG \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} d\nu + \begin{bmatrix} B & -L \\ B & 0 \end{bmatrix}\begin{bmatrix} d\beta \\ d\eta \end{bmatrix}\end{aligned} \qquad (11)$$

with the output $e = x - \hat{x}$.

The H_2 norm of the above system is $Tr\left(C^T PC\right)$ where $C = [I \quad 0]$ and

$$P = \begin{bmatrix} X & Z \\ Z^T & Y \end{bmatrix} > 0$$

is the solution of the Riccati-type equation

$$\begin{aligned}&\begin{bmatrix} A-LC & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} X & Z \\ Z^T & Y \end{bmatrix} + \begin{bmatrix} X & Z \\ Z^T & Y \end{bmatrix}\begin{bmatrix} A-LC & 0 \\ 0 & A \end{bmatrix}^T \\ &+ \begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix}\begin{bmatrix} X & Z \\ Z^T & Y \end{bmatrix}\begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix}^T \\ &+ \begin{bmatrix} 0 & -LG \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X & Z \\ Z^T & Y \end{bmatrix}\begin{bmatrix} 0 & -LG \\ 0 & 0 \end{bmatrix}^T \\ &+ \begin{bmatrix} B & -L \\ B & 0 \end{bmatrix}\begin{bmatrix} B & -L \\ B & 0 \end{bmatrix}^T = 0\end{aligned} \qquad (12)$$

The block (1,1) of the above equation gives:

$$\begin{aligned}&AX + XA^T - XC^T K^{-1}CX + BB^T + DYD^T \\ &\quad + \left(L - XC^T K^{-1}\right)K\left(L - XC^T K^{-1}\right)^T = 0\end{aligned} \qquad (13)$$

where K is defined by (10), and the block (2,2) coincides with (9).

Since the H_2 norm of the system (11) is $Tr(X)$, from Propositions 3 and 4 it follows that this is minimal if the last term in the left hand side of (13) vanishes, namely if L has the expression (7) with X being the stabilizing solution of (8) and thus the proof ends.

Remark 2. The optimal filter gain L given by (7) is computed solving first the equation (9) (which has a solution $Y \geq 0$ since the system (1) was assumed to be ESMS) and then the Riccati equation (8). In order to solve the Lyapunov-type equation (9) one can use the iterative procedure:

$$AY_{k+1} + Y_{k+1}A^T + DY_k D^T + BB^T = 0,\ k = 0,1,\ldots$$

with $Y_0 = 0$. The proof of the convergence of sequence $Y_k \geq 0$ may be found in a slightly modified version in (Dragan *et al*, 1997).

Remark 3. In the absence of the state-dependent noise in both equations (1), namely if $D = 0$ and $G = 0$, the result stated in Theorem 1 simply reduces to the classical Kalman-Bucy filter.

4. A NUMERICAL EXAMPLE

In order to illustrate the theoretical results derived in the previous section one considers the stochastic system with multiplicative noise (1) with

$$A=\begin{bmatrix}0 & 1\\ -1 & -0.4\end{bmatrix},\ D=\begin{bmatrix}0.5 & 0.3\\ 0.3 & -0.12\end{bmatrix},\ B=\begin{bmatrix}-1\\ 1\end{bmatrix},$$

$$C=\begin{bmatrix}-0.5 & 1\\ 0.25 & 0.5\end{bmatrix},\ G=\begin{bmatrix}0.25 & 0.5\\ 0.5 & -0.75\end{bmatrix}$$

Using Theorem 1 and Remark 2 one obtains the optimal gain

$$L=\begin{bmatrix}-0.8211 & -0.1219\\ 0.8649 & 0.2848\end{bmatrix}$$

for which the eigenvalues of $A-LC$ are $\{-0.8937 \pm 0.9687j\}$ showing thus that the obtained observer is stable.

In Figure 1 the following time responses are plotted: in (a) the true states of the stochastic system (1), in (b) the measured outputs of (1) and finally in (c), the outputs of the observer (6). Analyzing the plots in Figure 1a and Figure 1c one can see that they are very similar which fact indicates very good estimation properties of the resulting observer.

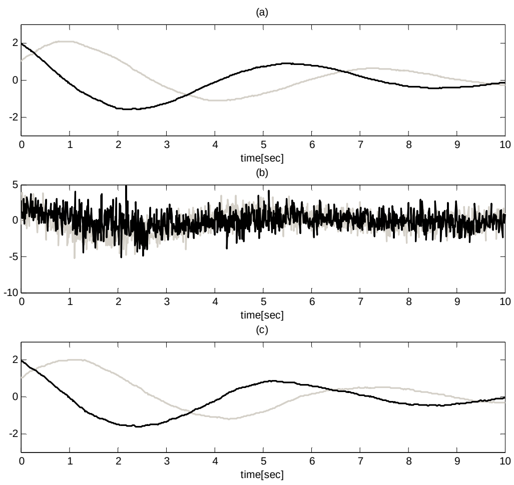

Fig. 1. Time responses: (a)-true states, (b)-measured outputs, (c)-estimated states

5. CONCLUSIONS

The present paper treats an estimation problem for linear stochastic systems corrupted with both additive white noise and with state-dependent noise. The considered structure of the observer is a deterministic Luenberger-type one which gain is determined by minimizing the H_2 norm associated with the mapping from the exogenous additive white noises to the estimation error. It is shown that the optimal gain depends on the solution of a system of algebraic Riccati and Lyapunov equations. In the particular case when the state-dependent noise is missing this system simply reduces to the well-known filtering Riccati equation from the Kalman-Bucy filtering theory. A numerical example illustrates the theoretical developments.

REFERENCES

Aström, K.J. (1970). *Introduction to stochastic control theory*, Math. in Science and Eng., Academic Press.

Dragan, V., A. Halanay and T. Morozan (1992). Optimal stabilizing controller for linear systems with state-dependent noise. *Stochastic analysis and applications*, **10**, 557-572.

Dragan, V., A. Halanay and A. Stoica (1997). A small gain theorem for linear stochastic systems. *Systems and Control Letters*, **30**, 243-251.

Hasminskii, R.Z. (1980). *Stochastic stability of differential equations*. Sijthoff and Noordhoff, Alpen aan den Rijn.

Kailath, T. (1974). A view of three decades of linear filtering theory. *IEEE Transactions on Information Theory*, **20**, 146-181.

Kalman, R.E. (1960). A new approach to linear filtering and prediction problems. *ASME Trans.-Part D, J. Basic Engineering*, **82**, 34-45.

Kalman, R.E. and R.S. Bucy (1961). New results in linear filtering and prediction theory. *ASME Trans.-Part D, J. Basic Engineering*, **83**, 95-108.

Kalman, R.E. (1963). New Methods of Wiener Filtering Theory. Proc. 1st Symp. Engineering Applications of Random Functions Theory and Probability, J.L. Bogdanoff and F. Kozin, Eds., Wiley, New-York, 270-388.

Mangoubi, R.S. (1998). *Robust estimation and failure detection*. Springer.

Stoica, A.-M. and D. Tiba (2007). A Kalman-filtering problem in the presence of multiplicative white noise. Eurogen Conference, Jyvaskyla, Finland, 11-13 June 2007.

Wiemmer, H.K. (1985). Monotonicity of maximal solutions of algebraic Riccati equations. *Systems and Control Letters*, **5**, 317-319.

Wonham, W.M. (1970). Random differential equations in control theory. *Probabilistic methods in applied mathematics*, **2**, 131-212, Academic Press.