BRANCHED MANIFOLDS, KNOTTED SURFACES AND DYNAMICAL SYSTEMS

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Abstract: The main goal of this paper is to introduce part of a new approach of proving the existence of a nontrivial knot on any embedded template. This proof in branched 2-manifold case, independent of Bennequin's inequality[Ghrist, 1997], which we make by first showing our simple template contains a nontrivial knot and then showing such a template can be contained in all templates that meet certain conditions as a subtemplate, enables us to generalize it later in this paper to certain forms of *3-template* in four dimensional dynamical systems by simply using the device of 'spinning' the knot with the corresponding lower dimensional template to obtain the spun knotted surface.

Keywords: Knots; branched manifolds; templates; knotted surfaces

1. INTRODUCTION

The theory of branched manifolds and templates in 3-dimensional systems is well-known and is given in details in[Ghrist, 1997]. One of the main results is that any embedded template contains a nontrivial knotted orbit. The proof depends on the inequality of Bennequin[Ghrist, 1997], which relates the genus of the Seifert surface spanning a link to its crossings in a closed braid diagram. This result is not available for higher-dimensional knots and branched manifolds, so it is important to find a proof that any embedded template contains a nontrivial knot which is independent of this inequality. We shall give part of such a proof in this paper and show how it may be generalized to 3-dimensional embedded templates in 4-dimensional dynamical systems and gives rise to the knotted surface. The proof will be based on showing that any embedded template contains at least one of a simple set of subtemplates which can be shown to contain nontrivial knots.

The paper is organized as follows. In section 2 we give a brief overview of branched manifolds and templates, and define the basic template we study in the following section. In section 3 we recall the basic template theory along with the introduction of ordering of orbits and adjacent orbits and then show that any template contains our basic template $\mathcal{T}(\tau_1, \tau_2)$ (for the notation, see the body of the paper) as a subtemplate contains a nontrivial knot. Based on the result in section 4, we try to generalize the case to 3-dimensional embedded templates in the final section by spinning the basic template and corresponding knots.

2. BRANCHED MANIFOLDS AND TEMPLATES

The main result of this paper is to generalize the theorem of ([Ghrist, 1997]) that any branched 2-manifold contains a knot to branch 3-manifolds.

We therefore first give a brief introduction to the theory of branched 2-manifolds and dynamical systems.

Definition 1. A branched two-manifold is a topological space which is locally a two-manifold or is locally a branched chart of the form in Fig.1.

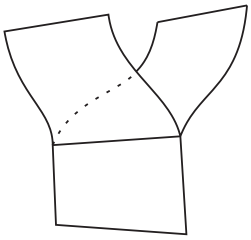
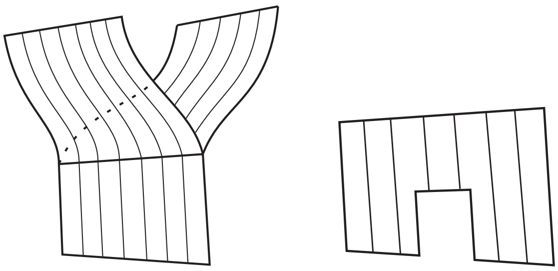


Fig. 1. A branched 2-chart

Definition 2. A 2-template is a compact branched 2-manifold with boundary and a smooth expansive semiflow built locally from two types of charts of the form in Fig.2.



(a) (b)

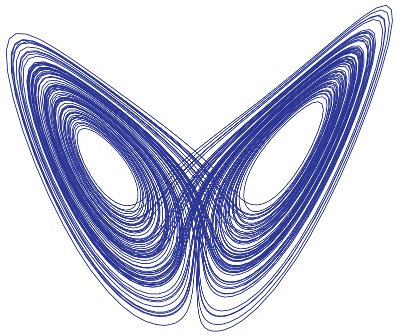
Fig. 2. (a) A Joining Chart , (b) A Splitting Chart

The importance of templates lies in the fact that for a flow ϕ_t on a 3-manifold M with hyperbolic chain-recurrent set, the link of periodic orbits of ϕ_t is in one-to-one correspondence with the link of periodic orbits on some embedded template in M. This is given on any finite sublink by ambient isotopy. (This is the template theorem of [Birman & Williams, 1983].)

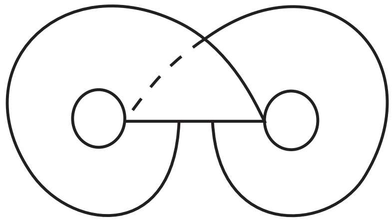
Example 1. Consider the Lorenz system

$$\dot{x_1} = 10(x_2 - x_1)$$
$$\dot{x_2} = (25 - x_3)x_1 - x_2$$
$$\dot{x_3} = x_1 x_2 - \frac{8}{3}x_3$$

The well-known Lorenz system and the template associated system is shown in Fig.3. The Lorenz attractor is a vector field in $\mathbb{R}^3$ which has an attractor with a dense set of hyperbolic periodic orbits and one hyperbolic singularity. It is thus of considerable importance and interest to study the template since such a 'template model' encodes all topological properties of the unstable periodic solutions of the dynamical system embedded within the attractor.



(a)



(b)

Fig. 3. (a) The Lorenz System, (b) The Associated Lorenz Template

3. SIMPLE PROOF OF TEMPLATE THEOREM FOR TEMPLATES CONTAINING $\mathcal{T}(0,0)$

(All the results here are proved in[Ghrist, 1997].)

Given any two-dimensional template, we can put a simple template $\mathcal{T}(0,0)$ consisting of just one joining chart and one splitting chart in a normal form then propagating the gaps backwards, as in Fig.4

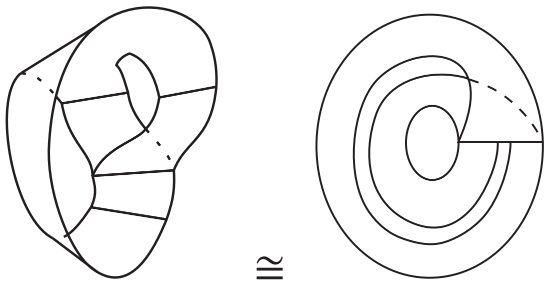


Fig. 4. Normal form of $\mathcal{T}(0,0)$

In the normal form the template is made up of a finite number of strips, which are topologically rectangles, which are jointed at two opposite ends to a branch line. To develop a symbolic language for templates we label the strips $x_1, \cdots, x_N$. The first important result is that the set of all forward orbits $\sum_{\mathcal{T}}$ which remain on the template $\mathcal{T}$ is given by the subshift of finite type given by the matrix $\mathcal{A}_{\mathcal{T}}$ where

$$\mathcal{A}_{\mathcal{T}}(i,j) = \begin{cases} 1 & \text{if incoming } x_i \text{ meets outgoing} \\ & x_j \text{ at a branch line,} \\ 0 & \text{otherwise.} \end{cases}$$

It is important to determine an ordering of the orbits on $\sum_{\mathcal{T}}$. For an orientable template, we first order the segments leaving each branch line as in Fig.5

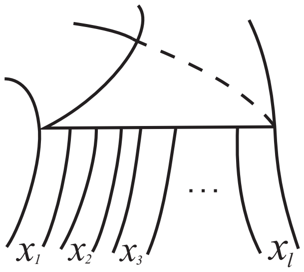


Fig. 5. Ordering the strips from a branch line: $x_1 \lhd x_2 \lhd \cdots \lhd x_l$ (or $x_l \lhd x_{l-1} \lhd \cdots \lhd x_1$)

Now order the orbits leaving a branch line lexicographically, i.e. if **a** and **b** leave the same branch line we define the order $\mathbf{a} \lhd \mathbf{b}$ if

$$\begin{aligned} \mathbf{a} &= x_{a_1} x_{a_2} \cdots x_{a_m} \cdots \\ \mathbf{b} &= x_{b_1} x_{b_2} \cdots x_{b_m} \cdots \end{aligned}$$

and $\quad x_{a_i} = x_{b_i}, 1 \leq i \leq k$ for some k
and $\quad x_{a_{k+1}} \lhd x_{b_{k+1}}$

in the ordering of Fig.5. For nonorientable templates we reverse the above ordering for any orbits belonging to sequences of strips with odd numbers of half-twists. (For general kneading theory, see[Milnor & Thurston, 1988].) Let σ denote the shift operator, i.e. if

$$\mathbf{a} = x_1 x_2 x_3 \cdots$$

then

$$\sigma \mathbf{a} = x_2 x_3 x_4 \cdots$$

If $\mathbf{u}$ and $\mathbf{v}$ begin on the same branch line and $\mathbf{u} \lhd \mathbf{v}$, we define the interval $(\mathbf{u}, \mathbf{v})$ to be the points $\mathbf{x}$ (beginning on the same branch line) such that $\mathbf{u} \lhd \mathbf{x} \lhd \mathbf{v}$. Then $\mathbf{u}$ and $\mathbf{v}$ are called *adjacent* if

$$\{\sigma^k \mathbf{u}\}_{k>0} \cap (\mathbf{u}, \mathbf{v}) = \{\sigma^k \mathbf{v}\}_{k>0} \cap (\mathbf{u}, \mathbf{v}) = \emptyset$$

We now prove that any template that contains $\mathcal{T}(0,0)$ as a subtemplate contains a nontrivial knot, independent of Bennequin's inequality. The crucial result is that the template $\mathcal{T}(0,0)$ is a subtemplate of every template which has no twists in any of the x_j strips.

Theorem 3. Every template $\mathcal{T}$ that contains $\mathcal{T}(0,0)$ contains a nontrivial knot.

We shall first consider the periodic solution on $\mathcal{T}(0,0)$ shown in Fig.6, and show that there is a nontrivial knot in it.

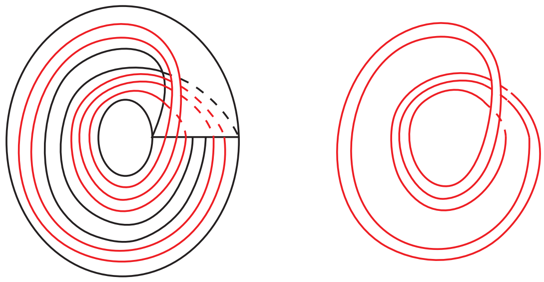


Fig. 6. A nontrivial knot on $\mathcal{T}(0,0)$

By a sequence of elementary moves, this is clearly a knot.(see Fig.7)

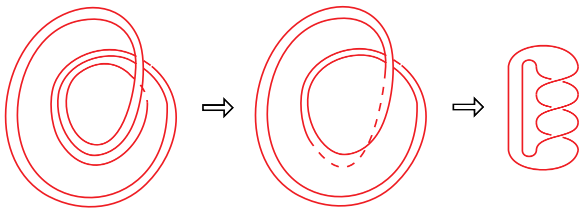


Fig. 7. Elementary moves on the knot

The consider any template $\mathcal{T}$ and let $\mathbf{a}, \mathbf{b}$ be nontrivial distinct words which are adjacent, and suppose that

$$\begin{aligned} \mathbf{a} &= x_1 x_2 \cdots x_k \\ \mathbf{b} &= x_{i_1} x_{i_2} \cdots x_{i_l} \end{aligned}$$

where the x_j's label the strips of $\mathcal{T}$ and contain no twists in any of them. Then $\mathbf{a}^\infty, \mathbf{b}^\infty \in \mathcal{T}$, and we consider the new symbols

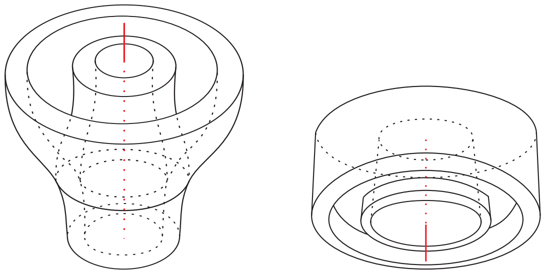
$$\begin{aligned} y_1 &= x_1 x_2 \cdots x_k \\ y_2 &= x_{i_1} x_{i_2} \cdots x_{i_l} \end{aligned}$$

Then clearly all words in y_1, y_2 lie on $\mathcal{T}(0,0)$.

Remark 4. This proof is done by showing that every template that contains $\mathcal{T}(0,0)$ as a subtemplate contains a nontrivial knot, which is still very specific. Similar proofs are also tenable for cases in which we can show the existence of nontrivial knots in basic templates of other forms(with twists). Nevertheless, it will be interesting and useful to give a complete proof for the more general case of $\mathcal{T}(\tau_1, \tau_2)$ which has an arbitrary number of twists on both strips of this refined basic template. We shall discuss this in a future paper and further generalize it in more general higher dimensional systems.

4. FOUR-DIMENSIONAL SYSTEMS

We now have a proof that two-dimensional templates contain $\mathcal{T}(0,0)$ have a nontrivial knotted orbit which is independent of Bennequin's inequality. It is therefore possible to generalize it to higher dimensions. In this paper we shall consider the three-dimensional case of branched manifolds in four-dimensional dynamical systems. The technical device we use is that of 'spinning' (see[Rolfsen, 2003]). Thus we will define a *3-template* to be given by gluing spun versions of the two elementary charts in Fig.2. These will appear as in Fig.8.



(a) (b)

Fig. 8. (a) Joining and (b) Splitting Chart in 3-dimensional Branched Manifolds

Note that the branching now takes place over an annulus and by pulling the splitting annulus back to the branching ones, we can put the 3-template in a normal form as in the 2-template case, shown Fig.9.

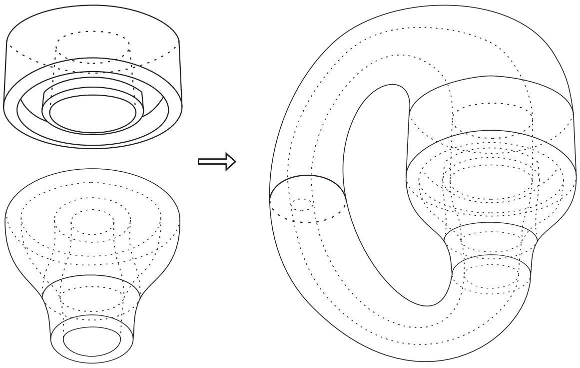


Fig. 9. The Spun Basic Template

The strips of 2-template become 'thick' cylinders which we will call tubes in the case of 3-templates and 1-dimensional orbits become knotted surfaces. We can number the cylinders $x_1, \cdots, x_N$ as in the case of strips, since there is a one-one map between them (this is only true in the spun case, of course). Strings in the x_i's then correspond to invariant surfaces in the dynamics and the theory is just the same as in the two-dimensional case. Hence by spinning the knots in the proof of Theorem 3, we can deduce immediately

Theorem 5. Any 3-template embedded in a 4-manifold which consists of local splitting and joining charts spun from $\mathcal{T}(0,0)$ without twists of the form in Fig.8 has a nontrivial knotted surface.

Remark 6. Since this theorem is deduced from the case of $\mathcal{T}(0,0)$ in the 2-template situation, it will also be our future work to generalize other forms of this with twists and consider the topological structure of the more general basic *3-template* in the four-dimensional system.

5. CONCLUSION

In this paper we have given another approach to partially prove theorem 3.1.13 of([Ghrist, 1997]) (which guaranntees a nontrivial knot on any embedded template), which is independent of Bennequin's inequality. This was done by showing that the simple template consisting of a pair of charts - one joining chart and one splitting chart can be contained as a subtemplate in template of certain forms and then showing that this simple template contains a nontrivial knot. The advantage of this proof is that it generalizes to certain four-dimensional manifolds containing 3-dimensional templates. Here we have considered only spun versions of the simple 2-dimensional template, for which the kneading theory generalizes in a direct way.

In the future we will extend the result of limited number of basic templates which can be contained in other templates as subtemplates to a complete coverage of all the basic templates consisting of only one joining and splitting chart(or of various twists forms) and the result of 3-dimensional templates which are not simply spun versions of the 2-dimensional case. This will require a more sophisticated kneading theory to give an order relation on the branch manifolds. The branched manifold theory will also be connected with higher dimensional chaotic systems.

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