STABILITY CRITERION FOR HYBRID SYSTEMS WITH DELAY

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Abstract: This paper presents hybrid systems with delay and one criterion for stability of these. Using time-domain technique, I use the Lyapunov aspect involved in stability. The paper presents a Lyapunov-Razumikhin stability result emphasizes characteristic of Lyapunov-Razumikhin functions.

Keywords: Hybrid systems, stability, delay, Lyapunov-Razumikhin functions

1. INTRODUCTION

Different aspects from real life can be modeled by hybrid systems. This kind of system has been known a great attention in the last decade because the application in many fields, a special attention giving to control. Generally speaking, a hybrid system is a mixture of continuous and discrete states. One characteristic of these is that in many cases the continuous states chancing by a discrete law.

The rader can read different aspects of the *hybrid systems* in *(*van der Schaft and Schumacher, 2000; Liberzon, 2003) or, like *piecewise smooth* dynamical systems name in (di Bernardo *et al.*, 2003). The examples can be drawn from a wide range of application areas, including process control (Lennartson *et al.*, 1996), constrained mechanical systems (Brogliato, 1999), robotics (Spong and Vidyasagar, 1989; Piiroinen, 2002), power systems (Hiskens, 2004), and power electronics (Yuan *et al.*, 1998). In fact, any physical device that exhibits hysteresis, or control loop with anti-wind-up limits (Goodwin *et al.*, 2001), is effectively a hybrid system. More details about hybrid systems and many references can be found in (DeCarlo *et al.*, 2000). The presence of the algorithms in hybrid systems gives efficiency those and there are many algorithms, see (Attia *et al.*, 2005), and the references from there.

There are many results on stability and stabilization in hybrid systems and some of these can be read in (DeCarlo *et al.*, 2000) but, however, there are few papers on switched (linear) systems with delay. In (Guangming *et al.*, 2002) is presented a controllability analysis for switched linear systems with delay in the control signal and in (Zhai *et al.*, 2003) is studied stability and L2 gain analysis for switching symmetric systems with time delay. Stability and stabilization of switched linear systems with state delay is studied in (Guangming and Wang, 2004). A variation in stability study is made using Lyapunov-Krasovskii functionals in (Balakrishnan *et al.*, 2004). In (Prajna and Jadbabaie, 2005) the Lyapunov-Razumikhin functions are used for safety verification of time-delay systems.

This paper is organized as follows. In Section 2 I present some notions, in Section 3 is presented the result of this paper and in Section 4 is the conclusion.

2. PRELIMINARIES

Different aspects of the hybrid systems are studying. We need one mathematical model for study and different notions and those are presented here.

2.1 Sources of delays

Modeling the hybrid systems using problems from real life, we expect to obtain some delays because only in one ideal model we don't have any influence of the physical characteristics of the components of the plant. We have:

-delays in transmission: the physical characteristics influence the transmission time. In area of computer and communications networks we must take into consideration when designing control laws the delays imposed by the communication network that we use to transmit/receive information for the industrial process under control. For example, a hybrid system operates over an industrial real-time communication network, usually named networked hybrid system, see (Davrazos and Koussoulas, 2002). This is a result of using a discrete event supervisory controller communicating with the local controllers through a communication network. As a result of this, the information, that the process exceeds some predefined limits in state variables, such as temperature, is perceived by the supervisor with a delay, which depends on the communication network. Sometimes there are systems with a lower bound on the delay time. Those can model communication or computational delays that prevent to make distinct, discrete actions in quick succession. The placement of such delay constraints is often used to prevent the synthesis of Zeno controllers which satisfy the safety property only by virtue of enforcing infinitely many events in finite time.

-sliding mode: the trajectory can come from a Ω_I subset of the state space and on switch set it can have a sliding mode for a time τ before leaves that switch set and goes into Ω_j subset as it can see in Fig. 1. In Fig. 2 we have a situation where the switch map is on the both sides of the switch set and we have a delay because the sliding mode on the switch set. Trying to model this behavior we can consider the next maps:

$$\Sigma : \tilde{x}_0 \mapsto x_1;\ f^+ : x_1 \mapsto \tilde{x}_1;$$
$$\Sigma : \tilde{x}_1 \mapsto x_0;\ f^- : x_0 \mapsto \tilde{x}_0$$

and the application is $(\Sigma \circ f^-) \circ (\Sigma \circ f^+)$. The map Σ could has the next form derived from sliding flow: $\Sigma(x) = e^{\tilde{A} \cdot \tau_S} \cdot x$, where $\tilde{A} = T \cdot A$, with T a term which depends on the form of the source system.

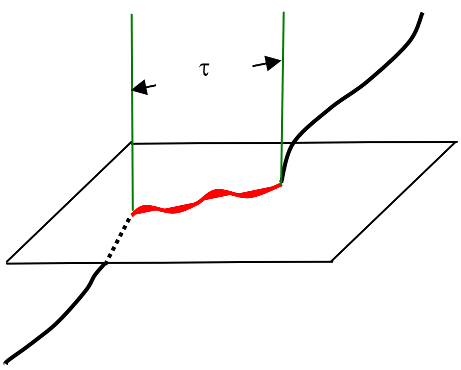

Fig. 1 The sliding mode for a trajectory has as effect a delay in state of the hybrid system with a value equals with time in sliding mode

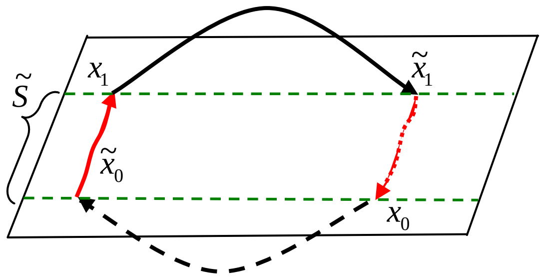

Fig. 2 One delay generates by the sliding mode. The switch maps are on the both sides of the switch set.

2.2 Mathematical model

One of the characteristics of the hybrid systems is that there is not a general model, but we have many models for different problems and applications. You can see more details about models in (DeCarlo *et al.*, 2000). In the general class of hybrid systems we consider delays on states obtaining the next mathematical model which has the τ value of the delay:

$$\begin{cases} \dot{x}(t) = f_{q(t)}(x(t)) + D_{q(t)} \cdot x(t-\tau) + \alpha(t), \\ x(t) = \varphi(t),\quad t \in [-\tau, 0],\quad x(0) = \varphi(0) = x_0 \\ q(t) = s(x(t), q(t^-)) \end{cases} \tag{1}$$

unde $\alpha(t) = 0$ sau $\alpha(t) = B_{q(t)} \cdot u(t)$. We can have $f_{q(t)}(x(t)) = A_{q(t)} x(t)$ for switched linear systems:

$$\begin{cases} \dot{x} = A_{q(i)} x(t) + D_{q(i)} \cdot x(t-\tau) \\ x(t) = \varphi(t),\quad t \in [-\tau, 0] \\ q(t) = s(x(t), q(t^-)) \end{cases} \tag{2}$$

which is a simplified form of (1).

2.3 Lyapunov-Razumikhin functions

The conditions for stability is often given by a function which has the property that is positive defined and it has the derivatives along its trajectories negative defined and its name is Lyapunov function.

A similar aspect of the stability is by using Lyapunov-Razumikhin functions. The Razumikhin functions means that one needs the negativity of the Lyapunov-Razumikhin candidate not for all trajectories, but only for some "critical" ones defined by the evolution of the system over one delay interval [*t*-τ, *t*]. In the linear case, if one assumes a simple quadratic Lyapunov candidate, this stability result is "strongly" related with the S - procedure of

Yakubovich, see (Yakubovich, 1977).

Let $C([-\tau, 0], \boldsymbol{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\boldsymbol{R}^n$ with the topology of uniform convergence. We have from (Niculescu, 2001):

Theorem 1 (Razumikhin Stability Theorem) Suppose that the function $f:\boldsymbol{R}\times C([-\tau, 0], \boldsymbol{R}^n) \to \boldsymbol{R}^n$ takes bounded sets of $C([-\tau, 0], \boldsymbol{R}^n)$ in bounded sets of $\boldsymbol{R}^n$ and suppose u; v; w: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ are continuous, nondecreasing functions such that $u(s)$, $v(s)$, $w(s)$ positive for $s > 0$, $u(0) = v(0) = 0$.
Assume that there exists a continuous function $V:\boldsymbol{R}\times\boldsymbol{R}^n \to \boldsymbol{R}^n$ such that:

$$u(\|x\|) \le V(t; x) \le v(\|x\|) \text{ for } t \in \boldsymbol{R},\ x \in \boldsymbol{R}^n.$$

The following statements hold:

a) $\dot{V}(t, x(t)) \le -w(\|x(t)\|)$ if $V(t+\theta, x(t+\theta)) < p(V(t, x(t)))$, $\forall\ \theta \in [-\tau, 0]$, then the trivial solution of $\dot{x}(t) = f(t, x_t),\ \ t \ge t_0$, with the initial condition $x_{t_0}(\theta) = \phi(\theta),\ \ \forall\,\theta \in [-\tau, 0]$, is uniformly stable.

b) If there exists a continuous nondecreasing function p: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$; $p(s) > s$, such that $\dot{V}(t, x(t)) \le -w(\|x(t)\|)$ if $V(t+\theta, x(t+\theta)) < p(V(t, x(t)))$, $\forall\ \theta \in [-\tau, 0]$, then the trivial solution of $\dot{x}(t) = f(t, x_t),\ \ t \ge t_0$, with the initial condition $x_{t_0}(\theta) = \phi(\theta),\ \ \forall\,\theta \in [-\tau, 0]$, is uniformly asymptotically stable.

If $u(s) \to +\infty$ as $s \to +\infty$, then the trivial solution is globally asymptotically stable.

2.4 Switch sets

Let Q be a discrete lot, $Q = \{1, 2, 3, \ldots, n_Q\}$ The function $s:H(=\boldsymbol{R}^n \times Q) \to Q$ by $s(x, i) = j$, where $x \in \boldsymbol{R}^n$ and $i,j \in Q$, means that we have *a switch* of discrete state from state i to state j. To put together all continuous states which are involved in this change, we define switch set. One image of the switch sets is in (Rubensson and Lennartson, 2000). In (Branicky, 1995) the notion of switch set is replaced by switch surface. The switch set $S_{i,j}$ is

$$S_{i,j} = \left\{x \in R^n \ \middle|\ s(x,i) = j\right\} \qquad (3)$$

For each $i \in Q$, the vector field $f(\cdot, i): R^n \to R^n$ is assumed to be locally Lipschitz continuous. This condition is necessary to assure some properties of the vector field. for the index i fixed.

The switch set can be given by switch functions. So, if a switch function is a map $s_{i,j}$: $\boldsymbol{R}^n \to \boldsymbol{R}^n$, then the switch set can be defined as $S_{i,j} = \{x \mid s_{i,j}(x) = 0\}$. Generally, the switch functions represent hyperplanes in the extended state space, i.e. $s_{i,j}(x) = C_{i,j}x + D_{i,j}$. Let assume that we have for our system m switch sets.

3. LYAPUNOV-RAZUMIKHIN STABILITY CRITERION IN HYBRYD SYSTEMS

The problem of stability for hybrid systems with delays is important in the context in which the delays for systems appear from different measures of technical characteristics of the plants. The stability results are extensions of Lyapunov theory where the existence of an abstract energy function satisfying certain properties verifies stability.

3.1 Stability criterion

Here I give one stability criterion for hybrid systems with delays. I use the characteristic of Lyapunov-Razumikhin functions and the main idea is that I impose stability conditions on trajectory which switch from one partition component of state space to another by switch sets on delay interval.
Let considerate a partition of state space $\bigcup_i \Omega_i = \boldsymbol{R}^n$, where for every Ω_i we have a V_i Lyapunov-Razumikhin function associated. We suppose u, v, w: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ are continuous, nondecreasing functions such that $u(s)$, $v(s)$, $w(s)$ positive for $s > 0$, $u(0) = v(0) = 0$.

Theorem 2. For a hybrid system with delay (1) we considerate that we have above partition and the $V_i:\boldsymbol{R}^+\times\boldsymbol{R}^n \to \boldsymbol{R}^n$ functions with the next characteristics:
a. $u(\|x\|) \le V_i(t, x) \le v(\|x\|)$, $t \in \boldsymbol{R}^+$, $x \in \Omega_i \subset \boldsymbol{R}^n$
b. there are the continuous nondecreasing functions p_i: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$, $p_i(s) > s$ such that $\dot{V}_i(t, x(t)) \le -w(\|x(t)\|)$ if $V_i(t+\theta, x(t+\theta)) < p_i(V_i(t, x(t)))$, $\forall\ \theta \in [-\tau, 0]$.
c. $V_j(x) \le V_i(x)$ if x switch from i state toward j state conform with S_{ij}.
Then the equilibrium point 0 is stable.

Proof: Trying to establish stability 0 equilibrum point in the sense of Lyapunov, it must be shown that for any $R > 0$ (any $R > 0$ such that the ball $B_R(0)$ is included in Ω_i) there exists $r(R) > 0$ such that $\|x_0\| < r$ implies that $\|x(t)\| < R$ for all $t \ge 0$. Let the initial state $(x_0, i) \in H_0$ be chosen such that $\|x_0\| < r$, where $H_0 \subset H$ is the subset of the initial states, $H = \boldsymbol{R}^n \times Q$ as in subsection *2.4*. Due to the first condition, and the continuity of the u, v functions, for any R > 0 there exists $r(R) > 0$ such that $v(r) < u(R)$ as it can be seen in Fig. 3. Let t_k denote the consecutive times when the trajectory passes from one region to another. Because τ is the length of the delay, we can split the positive axis in the intervals $[t_h-\tau, t_h]$, where $t_h = t_0 + h\cdot\tau$. For $h = 0$ and $t_0 = 0$, we have the interval $[-\tau,0]$. Let the trajectory starts from Ω_i for $t \geq t_0(=0)$ and which have the property $V_i(t+\theta, x(t+\theta)) < p_i(V_i(t, x(t)))$, $\forall\ \theta \in [-\tau, 0]$. If the trajectory moves from Ω_i to Ω_j on $[t_h-\tau, t_h]$ then the c. condition from theorem assure the decreasing of the Lyapunov function values or, with other words, of the energy. If $x(t) \in \Omega_i$ for $t \in (t_0, t_1)$ then $V(x(t)) = V_i(x(t))$, $t \in (t_0, t_1)$. By integration we obtain:

$$V(x(t)) \leq V(x(0)) - \int_0^t w(\|x(\tau)\|)d\tau\,, \qquad (4)$$

$t \in (0, t_1)$. If t_1 is infinite, then the trajectory never leaves Ω_i and solution is stable, using a. and b. properties conform Theorem 1, Razumikhin stability theorem from (Hale and Verduyn Lunel, 1993) or (Niculescu, 2001), so the conclusion. Otherwise, let the trajectory stays in Ω_j for $t \in (t_k, t_{k+1})$. Conform with the verification step from the mathematical induction method, we assume that we have (4) for $t \in (t_k, t_{k+1})$. We assume that the time for switch from Ω_j to another region Ω_h is t_{k+2}, so for $t \in (t_{k+1}, t_{k+2})$ the trajectory is inside of Ω_j. If we integrate on (t_{k+1}, t_{k+2}), we have

$$V(x(t)) \leq V(x(t_{k+1})) - \int_{t_{k+1}}^t w(\|x(\tau)\|)d\tau \qquad (5)$$

From the c. condition we have $V_j(x(t_{k+1}+\varepsilon)) \leq V_i(x(t_{k+1} - \varepsilon))$ with $\varepsilon > 0$, $\varepsilon \to 0$. So, using (5) and then (4) with $t = t_{k+1}$, results:

$$V(x(t)) \leq V(x(t_{k+1})) - \int_{t_{k+1}}^t w(\|x(\tau)\|)d\tau \leq$$

$$\leq V(x(0)) - \int_0^{t_{k+1}} w(\|x(\tau)\|)d\tau - \int_{t_{k+1}}^t w(\|x(\tau)\|)d\tau$$

$$= V(x(0)) - \int_0^t w(\|x(\tau)\|)d\tau \text{ for } t \in (t_{k+1}, t_{k+2}).$$

And from induction results that we have (4) for any $t \geq t_0(=0)$. We have w is positive and hence $V(x(t)) \leq V(x_0)$ for all $t \geq 0$. Finally, we have $u(\|x(t)\|) \leq V(x(t)) \leq V(x_0) \leq v(x_0) \leq v(r) \leq u(R)$ and result that $\|x(t)\| < R$ for all $t \geq 0$. In this way we have the condition for stability.

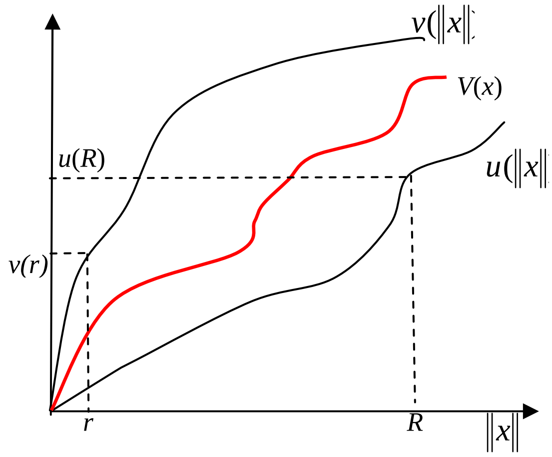

Fig. 3 The visualization of $v(R) < u(r)$. For any $R>0$ there is $r(R)>0$ with above relation

The stability characteristic using Lyapunov-Razumikhin and his utility is for a hybrid systems with delay given by that the trajectory fellow the condition only on delay interval $[-\tau, 0]$. On the other hand, we can replace $\dot{V}_i(t, x(t)) \leq -w(\|x(t)\|)$ with $\dot{V}_i(t, x(t)) \leq 0$.

4. CONCLUSIONS

The paper presents hybrid system with delay and one criterion for stability of this. Using time-domain technique, I use the Lyapunov aspect involved in stability. The paper presents a Lyapunov-Razumikhin stability result emphasizes characteristic of Lyapunov-Razumikhin functions. The extension stability to hybrid systems is naturally and it can be usefull.

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