OSCILLATIONS BY POINCARÉ MAPS IN HYBRID SYSTEMS

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Abstract: This work treats the oscillation aspect of the hybrid systems using Poincaré maps. I present the framework for Poincaré map and the main properties after that I define them for hybrid systems. I present a way to construct the Poincaré map for hybrid systems and properties of it for this kind of systems. I give steps for the limit cycle in a hybrid system. © *Copyright*

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1. INTRODUCTION

Many technical aspects from real life are modelled by a special class of systems, which are mixture of continuous and discrete states; in many cases the continuous states chancing by a discrete law.

Such systems have come to be known as *hybrid systems (*van der Schaft and Schumacher, 2000; Liberzon, 2003) or *piecewise smooth* dynamical systems (di Bernardo *et al.*, 2003). The examples can be drawn from a wide range of application areas, including process control (Lennartson *et al.*, 1996), constrained mechanical systems (Brogliato, 1999), robotics (Spong and Vidyasagar, 1989; Piiroinen, 2002), power systems (Hiskens, 2004), and power electronics (Rajaraman *et al.*, 1996; Yuan *et al.*, 1998). In fact, any physical device that exhibits hysteresis, or control loop with anti-wind-up limits (Goodwin *et al.*, 2001), is effectively a hybrid system. More details about hybrid systems and many references can be found in (DeCarlo *et al.*, 2000). One of the characteristic of the hybrid systems is that there are no a single model for a hybrid systems, and we have many models for different situations and problems. Another aspect is the presence of the algorithmic filed in hybrid systems, aspect which gives efficiency. There are many algorithms, see (Attia *et al*., 2005), where there are different references. You can see the link between Poincaré map and algorithms in (Asarin *et al.*, 2001).

2. PRELIMINARIES

Some notions and techniques need to be known here, and they come from the dynamical field like the flow. Also, the Poincaré map notion is presented and methods linked with it.

2.1 Trajectory sensitivities

The dynamical behaviour of continuous-time systems, such as power systems, can be expressed in terms of the *flow*,

$$x(t) = \phi(x_0; t) \qquad (1)$$

which describes the evolution of dynamic states x over time, starting from the initial condition $x(t_0) = x_0$. In general, ϕ cannot be expressed in closed form, and so must be obtained by numerical integration, see (Shampine, 1994). The flow may well describe behaviour that involves interactions between continuous dynamics and discrete events, as you can see (Hiskens, 2004).

Algorithms for locating limit cycles require the sensitivity of a trajectory (flow) to perturbations in initial conditions, as in (Stoer and Bulirsch, 1993). To obtain the sensitivity of the flow Á to initial conditions x_0, the Taylor series expansion of (1) is formed. Neglecting higher order terms gives

$$\Delta x(t) = \frac{\partial \varphi(t)}{\partial x_0}\Delta x_0 \equiv \Phi(x_0, t)\Delta x_0 \qquad (2)$$

where Φ is the *sensitivity transition matrix*, or *trajectory sensitivities*, associated with the *x* flow, see (Frank, 1978). The equation (2) describes the change $\Delta x(t)$ in a trajectory, at time *t* along the trajectory, for a given (small) change in initial conditions Δx_0. Space limitations preclude the inclusion of the variational equations describing the evolution of Φ. Full details are given in (Hiskens and Pai, 2000). It should be emphasized that Φ does not require smoothness of the underlying flow ϕ. Trajectory sensitivities are well defined for the non-smooth and/or discontinuous flows.

The computational burden of generating Φ is minimal. It is shown in (Li *et al.*, 2000).) that when an implicit numerical integration technique such as trapezoidal integration is used, trajectory sensitivities can be obtained as a by-product of computing the underlying trajectory.

2.2 Poincaré Map

Limit cycles and their stability can be determined using Poincaré maps, as it can be seen in (Seydel, 1994; Parker and Chua, 1989). The notion of Poincaré map comes from dynamical systems, but it can be extended to hybrid dynamical systems.

This Poincaré map notion uses the flow notion. A Poincaré map effectively samples the flow of a periodic system once every period. The concept is illustrated in Figure 1. If the limit cycle is stable, oscillations approach the limit cycle over time. The samples provided by the corresponding Poincaré map approach a fixed point. An unstable limit cycle results in divergent oscillations. For such a case the samples of the Poincaré map diverge.

To define a Poincaré map, consider the limit cycle Γ shown in Figure 1. Let Σ be a hyperplane transversal to Γ and defined by

$$\Sigma = \left\{ x : \sigma^T \cdot (x - \tilde{x}) = 0 \right\} \qquad (3)$$

where $\tilde{x}$ is a point anchoring Σ, and σ is a vector normal to Σ. The trajectory emanating from x^* will again encounter Σ at x^* after *T* seconds, where *T* is the minimum period of the limit cycle. Due to the continuity of the flow ϕ with respect to initial conditions, trajectories starting on Σ in a neighbourhood of x^* will, in approximately *T* seconds, intersect Σ in the vicinity of x^*. Hence ϕ andΣ define the Poincaré map

$$x_{k+1} = P(x_k) := \phi(x_k, \tau_r(x_k)) \qquad (4)$$

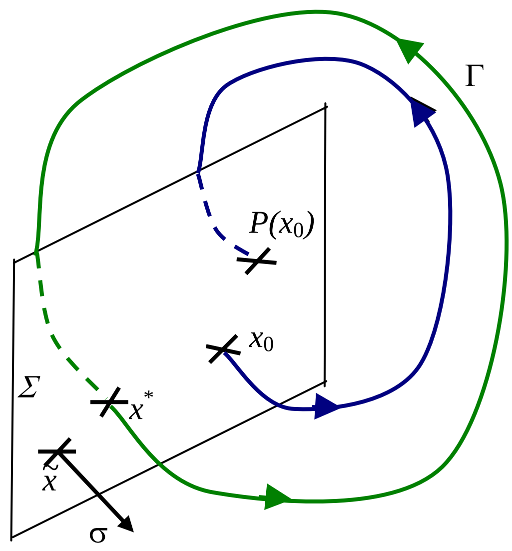

Fig. 1 Poincaré map and limit cycles for dynamical systems. We can see the both kind of behavior for the flow.

where $\tau_r(x_k) \approx T$ is the time taken for the trajectory to return to Σ. Another name for Poincaré map is first return map, and it is clear why. Complete details can be found in (Seydel, 1994; Parker and Chua, 1989).

2.3 Shooting method

From (4), it can be seen that a point x^* on the limit cycle can be located by using Newton's method to solve the nonlinear algebraic equations

$$F_l(x^*) = \phi(x^*, \tau_r(x^*)) - x^* = 0. \qquad (5)$$

We can define a iterative process for solve this problem. The solution process therefore has the iterative form

$$x^{i+1} = x^i - (DF_l(x^i))^{-1} \cdot F_l(x_i) \qquad (6)$$

It is shown in (Donde and Hiskens, 2006) that the Jacobian DF_l is given by

$$DF_l(x^i) = \left(I - \frac{f|_{\tau_r(x^i)}\sigma^T}{\sigma^T f|_{\tau_r(x^i)}} \right)\Phi(x^i, \tau_r(x^i)) - I \qquad (7)$$

where *f* is given by (2) and *I* is the *n*-dimensional identity matrix. Notice that because the flow ϕ and associated sensitivities Φ are well defined for non-smooth systems, solution of (5) is also well defined for such systems.

It can be seen from (5) that evaluation of $F_l(x^i)$ at each iteration requires numerical integration. This process is therefore referred to as a *shooting method* as it can be seen in (Stoer and Bulirsch, 1993).

2.4 Limit cycle stability

Stability of the Poincaré map (4) is determined by linearizing P at the fixed point x^*, i.e.,

$$\Delta x_{k+1} = DP(x^*)\Delta x_k \tag{8}$$

From the definition of $P(x)$ given by (4), it follows that

$$DP(x^*) = \left(I - \frac{f|_{x^*}\sigma^T}{\sigma^T f|_{x^*}} \right)\Phi(x^*, T) \tag{9}$$

where $\tau_r(x^*) = T$. The matrix $\Phi(x^*, T)$ is exactly the trajectory sensitivity matrix after one period of the limit cycle, i.e., starting from x^* and returning to x^*. This matrix is called the *monodromy matrix*. It is shown in (Parker and Chua, 1989) that for an autonomous system, one eigenvalue of $\Phi(x^*,T)$ is always 1, and the corresponding eigenvector lies along $f|_{x^*}$. The remaining eigenvalues of $\Phi(x^*, T)$ coincide with the eigenvalues of $DP(x^*)$, and they are known as the *characteristic* (or *Floquet*) *multipliers* m_i of the periodic solution. The characteristic multipliers are independent of the choice of cross-section Σ.

Because the characteristic multipliers m_i are the eigenvalues of the linear map $DP(x^*)$, they describe the (local) stability of the Poincaré map $P(x_k)$. Hence the (local) stability of the periodic solution is determined by:

- All m_i lie within the unit circle, i.e., $|m_i| < 1; \forall i$. The map is stable, so the periodic solution is stable.
- Some m_i lie outside the unit circle. The periodic solution is unstable.

Interestingly, there exists a particular cross-section Σ^*, such that

$$DP(x^*)\zeta = \Phi(x^*,T)\zeta \tag{10}$$

where $\zeta \in \Sigma^*$. This cross-section Σ^* is the hyperplane spanned by the n - 1 eigenvectors of $\Phi(x^*, T)$ that are not aligned with $f|_x$. Therefore the vector σ^* that is normal to Σ^* is the left eigenvector of $\Phi(x^*, T)$ corresponding to the eigenvalue 1. The hyperplane Σ^* is invariant under $\Phi(x^*, T)$, i.e., $\Phi(x^*, T)$ maps vectors $\zeta \in \Sigma^*$ back into Σ^*.

3. POINCARÉ MAPS IN HYBRYD SYSTEMS

The hybrid systems field has an big increase and many aspects can be treated by different approaches for solving problems from real life in many fields.

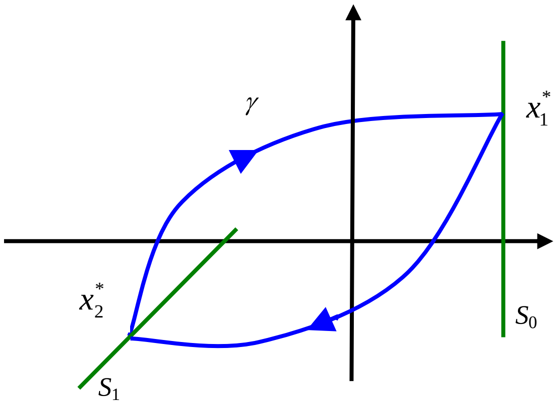

Fig. 2 The periodic solution γ of a second order hybrid system.

3.1 Switch sets

Let Q be a discrete lot, $Q = \{1, 2, 3, \ldots, n_Q\}$ The function $s{:}H(=\boldsymbol{R}^n \times Q) \rightarrow Q$ by $s(x, i) = j$, where $x \in \boldsymbol{R}^n$ and $i,j \in Q$, means that we have *a switch* of discrete state from state i to state j. To put together all continuous states which are involved in this change, we define switch set. One image of the switch sets is in (Rubensson and Lennartson, 2000). In (Branicky, 1995) the notion of switch set is replaced by switch surface.

The switch set $S_{i,j}$ is defined by

$$S_{i,j} = \{x \in R^n \mid s(x,i) = j\} \tag{11}$$

For each $i \in Q$, the vector field $f(\cdot, i): R^n \rightarrow R^n$ is assumed to be locally Lipschitz continuous.

The switch set can be given by switch functions. So, if a switch function is a map $s_{i,j}$: $\boldsymbol{R}^n \rightarrow \boldsymbol{R}^n$, then the switch set can be defined as $S_{i,j} = \{x \mid s_{i,j}(x) = 0\}$. Generally, the switch functions represent hyperplanes in the extended state space, i.e. $s_{i,j}(x) = C_{i,j}x + D_{i,j}$. Let assume that we have for our system m switch sets.

3.2 Limit cycles in hybrid systems

The study of the limit cycles in the hybrid systems is a natural aspect of the oscillation behavior and there are different approaches. In (Gonçalves, 2003) is presented a technique by calculation solutions of differential equations and imposing the condition for limit cycle ϕ. For every switch set S_{i+1} we have $\phi(t_1^* + t_2^* + \ldots + t_i^*) = x_{i+1}^* \in S_{i+1}$ and the cycle condition $\phi(t_1^* + t_2^* + \ldots + t_k^*) = x_{k+1}^* = x_1^* \in S_{k+1}$. Here t_i^* represents the time when the trajectory of the system intersects one switch set and change his behavior. The period of the ϕ cycle is $t^* = t_1^* + t_2^* + \ldots + t_k^*$. The initial condition is $\phi(t_1^*)$ = x_1^* .for the general hybrid system.

An image of the limit cycle for a linear system in $\mathbf{R}^2$ is presented in (Gonçalves, 2003) for $\dot{x} = A_i x + B_i$:

$$A_1 = A_2 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix},\ B_1 = \begin{pmatrix} -3 \\ -2 \end{pmatrix},\ B_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

The initial condition is $\phi(1.24)$ = $(1\ \ 0.87)^{/}$. The switching rule with memory that uses system 1 until the trajectory intersects the switching surface S_1, and then uses system 2 until the trajectory intersects the switching surface 2, and so on. The switching surface are given by $C_{1,2}$ = (-1 1), $D_{1,2}$ = -1, $C_{2,1}$ = (1 0), $D_{2,1}$ = -1. After solving the conditions, the results are: t_1^* = 1.24, t_2^* = 1.35, $x_1^* = x_3^*$ = $(1\ 0.87)^{/}$, x_2^* = $(-1.84\ -0.84)^{/}$ with the image in Figure 2.

Local stability of limit cycles can be checked by linearizing the Poincaré map, but global stability or characterization of stability in regions around limit cycles cannot be checked or found. The Lyapunov functions cannot be constructed in the state space to prove stability of limit cycles. Gonçalves (2003) presents the impact map notion, for that he parameterize the switching surfaces using the orthogonal complement, and gives his role in the locally stable limit cycle. In (Gonçalves *et al.*, 2003) he constructs a new notion, namely Lyapunov functions on switching surfaces, for study limit cycles.

3.3 Transversal section for Poincaré map

Construction of transversal set is made in the next way: the cross section $\Sigma \subset \mathbf{R}^n$, of dimension n-1, need not to be planar, but must be chosen so that the flow is everywhere transverse to it, that is $f(x)\cdot n(x) \neq 0$ for all $x \in \Sigma$, where $n(x)$ is the unit normal to Σ at x. We can take the tangent on flow in the intersection point between flow and switch set and after the normal on the tangent, as it can see in Figure 3.

For the hybrid systems we have to take for every switch set a surface which is perpendicularly on the tangent surface in the point where the trajectory

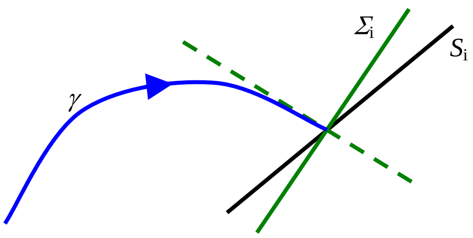

Fig. 3 We take the normal on the tangent in intersection between switch set and the flow.

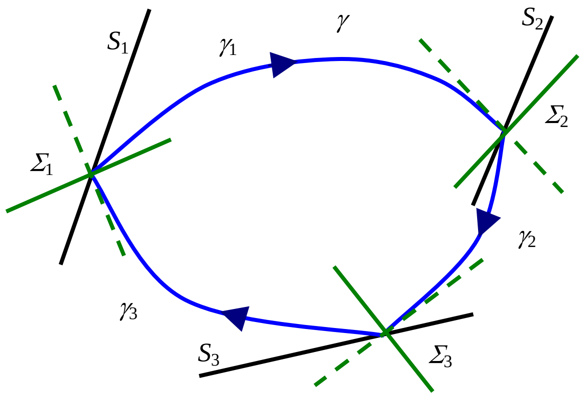

Fig. 4 For every intersection point we construct the transversal hyperplane taking the tangent to flow in that point and after constructing the normal on it.

intersects the switch set, as in Figure 4. In this way we obtain the Poincaré map for the hybrid system. If $p \in \gamma$, and Σ, with $p \in \Sigma$, is a cross section than if ϕ is transverse to Σ, we have $T_x(\Sigma)$ the tangent space in x. We can considerate a base on tangent space, which is a vector space, and after apply representations in tangent space using the base of it. In the next, we can apply a map between the two tangent spaces and uses the representation in bases.

Theorem 1. The stability of one fix point for the hybrid Poincaré map reflects the stability of the limit cycle γ for the compose flow ϕ.

Proof: We can considerate one function between every two pair of transversal sets and after we compose that functions. For three transversal sets, let say Σ_i, Σ_j and Σ_k, we have two function between them, let say it P_{ij}, respectively P_{jk}, and the condition $P_{ij}(x_0)$ must be in an open set included in Σ_j for $x_0 \in \Sigma_i$. We'll have $(P_{jk} \circ P_{ij})(x_0)$ belongs to Σ_k, and so $P_{jk} \circ P_{ij}$ can be seen as a map from Σ_i to Σ_k. So, for P_{12}, P_{23}, …, $P_{n-1\ n}$ and P_{n1} if we have $(P_{n1} \circ P_{n-1\ n} \circ \ldots \circ P_{23} \circ P_{12})(p) = p \in \Sigma_1$ then we have a limit cycle for the hybrid system.

In the theory of standard Poincaré map we have $P{:}U \rightarrow \Sigma$, $P(q) = \phi_t(q)$ for $q \in U$, where $U \subseteq \Sigma$ is a neighborhood of p. In the hybrid case $P = P_{n1} \circ P_{n-1\ n} \circ \ldots \circ P_{23} \circ P_{12}$. Let consider the limit cycle γ as composed from γ_1, γ_2, … , γ_n, where every γ_i corresponding to a part of the composed map, namely $P_{i\ i+1}$. We consider $P_{i\ i+1}(q_i) = \phi_i(q_i)$, where $q_i \in U_i \subseteq \Sigma_i$. But γ_i is a part of the limit cycle γ and γ_i corresponding to ϕ_i. In this way we have a direct corresponding between P and ϕ, $\phi = \phi_n \circ \ldots \circ \phi_1$.

Theorem 2. The shooting method can be applied to establish the limit cycle into a hybrid system.

Proof: Indeed, at every step we have to calculate points on the transversal sets and the iterative process is well-defined by the point x_i on Σ_i. The compose flow ϕ is determined by corresponding flows ϕ_i between switch sets. The (5) now becomes:

$$(\phi_n \circ \phi_{n-1} \circ ... \circ \phi_2 \circ \phi_1)(x^*, \tau_r(x^*)) - x^* = 0. \quad (12)$$

where $x^* \in \Sigma_1$.

4. CONCLUSIONS

The dynamical behavior of a Poincaré map for hybrid systems is a tool for studying limit cycles. I present the framework for Poincaré map and the main properties after that I define it for hybrid systems. I present a way to construct the Poincaré map for hybrid systems and properties of it for this kind of systems.

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