TIME VARIANT NONLINEAR SYSTEM IDENTIFICATION BY DISTRIBUTIONS; APPLICATION FOR WASTEWATER BIODEGRADATION PROCESS

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Abstract: This paper presents a procedure for time variant nonlinear system identification based on distribution theory. Some of the system parameters change in time according to unknown laws. These laws are expressed as finite degree time polynomials whose parameters are included in the set of parameters to be identified. Mainly it is an extension of the procedure developed in (Marin et al., 2005). Considering functionals weighted by polynomials, it is possible to transform a time variant differential system of equations to an algebraic system in unknown parameters. The hierarchical structure identification method for rational expressions, proposed in (Marin et al., 2005), is now extended to time variant systems. An application for parameter identification of a wastewater biodegradation process, considering time variant yield coefficients is presented.

Keywords: Identification; Distribution theory; Functionals, Bioprocesses.

1. INTRODUCTION

As is presented in (Marin et al., 2005), progresses have been made in the area of continuous-time system identification. Many discussions, methods and results on continuous-time identification are presented in (Unbehauen and Rao, 1987), (Li and Billings, 2001), (Landau et al., 2001), (Sinh and Rao, 1991), (Marin, 1992, 1993, 1999); (Bastogne et al., 1997); (Hoverkamp et al., 1996); (Overschee and De Moor, 1996).

A novel approach for continuous-time system identification is that based on distribution theory, using deterministic distributions (Marin, 2002) or random distributions (Ohsumi et al., 2002). Identification of the non-linear continuous-time systems is far away more complicated. The traditional procedures are based on the Volterra functional series (Schetzen, 1980), expressed in time domain (Boyd and Chua, 1985) or frequency domain (Li and Billings, 2001).

The parameter identification of deterministic nonlinear continuous-time systems (NCTS), modeled by polynomial type differential equation, has been considered by numerous authors, (Pearson and Lee, 1985), (Patra and Unbehauen, 1995). In (Marin et al., 2005), it is presented a method for identification of nonlinear continuous time systems (NCTS) considering that the unknown parameters can appear in rational relations with measured variables. Using techniques utilized in distribution approach (Marin 1992, 1993, 1999), the measurable functions and their derivatives are represented by functionals on a fundamental space of testing functions. Such systems are common in biotechnology (Bastin and Dochain, 1990), (Petre, 2002), (Selişteanu et al., 2004).

The main idea from (Marin et al., 2005) is to use a hierarchical structure of identification. First, some state equations are utilized to obtain a set of linear equations in some parameters. The results of this first stage of identification are utilized for expressing other parameters by linear equations. This process is repeated until all parameters are identified.

In the present paper, the above idea of hierarchical identification has been extended to time variant systems.
Variable parameters are modeled by finite degree time polynomials whose unknown coefficients are included in the set of parameters to be identified.

To transform a differential time variant system of equations to an algebraic system of functionals, the so-called weighted distributions are considered. Weighted distributions are nothing else rather the product between time functions and distributions.

The paper is organized as follows: The structure of a NCTS describing a wastewater biodegradation process is given in Section 2. Section 3, presents some aspects regarding weighted distributions and their derivatives. The problem statement of continuous time variant system identification is analyzed in Section 4, followed, in Section 5, by presenting the distribution approach of identification. The hierarchical structure of identification and estimation equations takes the space of Section 6. Some experimental results are presented in Section 7, and conclusions in Section 8.

2. MATHEMATICAL MODEL OF WASTE WATER BIODEGRADATION PROCESS

Shall we consider, as in (Marin et al., 2005), the same system represented by the model of a wastewater treatment plant, but, assuming now that the values of some unknown parameters can modify in time. This is a biomethanation process of wastewater biodegradation, producing methane gas. Such reactions take place inside a continuous stirred tank bioreactors whose reduced mathematical models express two phases processes as presented in (Selișteanu et al., 2004), (Bastin and Dochain, 1990), (Petre, 2002). The following simplified reaction scheme is considered,

\[ S_1 \rightarrow X_1 + S_2 ; \quad S_2 \rightarrow X_2 + P_1 , \]

where: \( S_1 \) represents the glucose substrate, \( S_2 \) the acetate substrate, \( X_1 \) is the acidogenic bacteria, \( X_2 \) the acetoclastic methanogenic bacteria and \( P_1 \) represents the product, i.e. the methane gas. The reaction rates are denoted by \( \phi_1, \phi_2 \).

The corresponding dynamical model, (Petre, 2002), is

\[ \frac{d}{dt} \begin{bmatrix} X_1 \\ S_1 \\ X_2 \\ S_2 \\ P_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -k_1 & 0 & 0 \\ -k_1 & 0 & 1 & 0 & 0 \\ k_2 & -k_1 & 0 & 1 & 0 \\ 0 & 1 & k_3 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ D \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ DS_m \\ 0 \end{bmatrix} \]

(2)

where the state vector of the model is

\[ \xi = [X_1, S_1, X_2, S_2, P_1]^T = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5]^T \]

(3)

whose components are concentrations in (g/l).

The parameters \( k_1, k_2, k_3, k_4 \) are the so-called yield coefficients.

In this paper, we consider that the first two yield coefficients change in time, that means,

\[ k_1 = k_1(t), \quad k_2 = k_2(t) \]

The reaction rates are nonlinear functions of the state components, expressed as

\[ \phi = \phi(\xi) = [\phi_1(\xi) \phi_2(\xi)]^T \]

(4)

The vector of feed rates and of rates of removal of components is denoted

\[ F = [0 \quad D \cdot S_m \quad 0 \quad 0 \quad -Q_i]^T \]

(5)

where, \( D \) is the dilution rate, a scalar in this particular case, \( S_m \) represents the concentration of the externally influent substrate–glucose, \( Q_i \) is the methane gas outflow rate.

The dynamical model (2) can be compactly written

\[ \frac{d\xi}{dt} = K(t) \cdot \phi(\xi) - D \cdot \xi + F . \]

(6)

In fact, this model describes the behavior of an entire class of biotechnological processes. It referees as the general dynamical state-space model of this class of bioprocesses (Bastin and Dochain, 1990). In (6), \( K(t) \) is the matrix of the yield coefficients \( k_y \)

\[ K(t) = \begin{bmatrix} 1 & -k_1(t) & 0 & k_3(t) & 0 \\ 0 & 1 & -k_1 & 0 & k_4 \end{bmatrix}^T \]

(7)

The reaction rates for this process are given by the Monod law

\[ \phi_1(\xi) = \mu_1 \cdot \frac{S_1 \cdot X_1}{K_{M_1} + S_1} , \]

(8)

and the Haldane kinetic model

\[ \phi_2(\xi) = \mu_2 \cdot \frac{S_1 \cdot X_2}{K_{M_2} + S_2 + S_2^2 / K_c} , \]

(9)

where \( K_{M_1}, K_{M_2} \) are Michaelis-Menten constants; \( \mu_1, \mu_2 \) represent specific growth rates coefficients and \( K_c \) is the inhibition constant.

Because variation speed of \( k_1(t) \), \( k_2(t) \) is not so high, their time law variation is locally approximated by straight lines

\[ k_1(t) = \theta_1 \cdot t + \theta_{10} , \quad \theta = [\theta_1, \theta_{10}] \]

(10)

\[ k_2(t) = \theta_2 \cdot t + \theta_{20} , \quad \theta = [\theta_2, \theta_{20}] \]

(11)

It is considered that all parameters \( \theta_1, \theta_{10}, \theta_2, \theta_{20} \) are constant during the time intervals when identification is performed. During these time intervals, different functionals representing distributions are evaluated to obtain a final result.

In such way, the time curves (10), (11) are piece wise approximated by straight lines.

For simplicity, all parameters characterizing the plant are gathered in a single vector

\[ \theta = [\theta_{11}, \theta_{10}, \theta_{20}, \theta_2, \theta_1, \theta_{10}, \theta_2, \theta_{20}]^T \]

(12)

where

\[ \theta_2 = k_2 \cdot \theta_2 ; \quad \theta_{20} = \theta_2 ; \quad \theta_1 = \mu_1 ; \quad \theta_{10} = \mu_{10} ; \]

(13)

\[ \theta_1 = K_{M_1} ; \quad \theta_2 = K_{M_2} ; \quad \theta_{20} = K_c \]

(14)

Because the dilution rate \( D \) can be externally modified, it will be considered as the third component of the input vector, \( u = [u_1, u_2, u_3]^T \).

The other two components of \( u \) are the concentration \( S_m \) and the methane gas outflow \( Q_i \).
3. WEIGHTED DISTRIBUTIONS AND THEIR DERIVATIVES

Let us denote by $\Phi$ the fundamental space from distribution theory (Kecs et al. 1975), of the real fundamental functions,

$$\varphi : \mathbb{R} \rightarrow \mathbb{R}, \ t \rightarrow \varphi(t),$$

(23)

with compact support $T$, having continuous derivatives at least up to the order $n$. The linear space $\Phi$ is organized as a topological space considering the norm,

$$\|\varphi\| := \sup_{0 \leq t \leq T} |\varphi^{(i)}(t)| = \max \{ \sup_{0 \leq t \leq T} |\varphi^{(i)}(t)| \}.$$ \(i \leq n\) \tag{24}

A distribution is a linear, continuous (in the above topology) real functional on $\Phi$,

$$F : \Phi \rightarrow \mathbb{R}, \varphi \rightarrow F(\varphi) \in \mathbb{R}$$ \tag{25}

Let

$$q : \mathbb{R} \rightarrow \mathbb{R}, t \rightarrow q(t)$$ \tag{26}

be a function that admits a Riemann integral on any compact interval $T$ from $\mathbb{R}$. Using this function, a unique distribution

$$F_q : \Phi \rightarrow \mathbb{R}, \varphi \rightarrow F_q(\varphi) \in \mathbb{R}$$ \tag{27}

can be built by the relation

$$F_q(\varphi) = \int_T q(t) \varphi(t) \, dt, \ \forall \varphi \in \Phi.$$ \tag{28}

Considering, at least, $q \in C^0(\mathbb{R})$, the following important equivalence take place (Barbu, 1985),

$$F_q(\varphi) = 0, \ \forall \varphi \in \Phi : q(t) = 0, \ \forall t \in \mathbb{R}. \tag{29}

Let

$$\rho_j : \mathbb{R} \rightarrow \mathbb{R}, t \rightarrow \rho_j(t), \ \rho_j \in C^0(\mathbb{R}), \ j \in J$$ \tag{30}

be a set of continuous time functions.

The product between a function (30), $\rho_j$, and a distribution (27), $F_q$, is a new distribution (Kecs et al. 1975),

$$F_q \rho_j : \Phi \rightarrow \mathbb{R}, \varphi \rightarrow F_q \rho_j(\varphi) \in \mathbb{R}, \ j \in J$$ \tag{31}

built through the relation

$$F_q \rho_j(\varphi) = \int_R \rho_j(t) \cdot q(t) \cdot \varphi(t) \, dt, \ \forall \varphi \in \Phi.$$ \tag{32}

Under this interpretation, a single function $q$ generates a family of distributions $F_q^j, \ j \in J$ on $\Phi$.

Let us denote by $\Phi^*$ the dual space of $\Phi$, which means the set of all distributions defined on $\Phi$.

Because $\varphi$ from (23), (24) has a finite compact support and all functions $\rho_j, \forall j \in J$ of (30) are continuous with derivatives at least up to the order $n$ then the functions

$$\varphi_{\rho_j} = \rho_j \cdot \varphi \in \Phi^*, \ \forall j \in J,$$ \tag{33}

are testing functions on $\Phi^*$, where

$$\varphi_{\rho_j}(t) = \rho_j(t) \cdot \varphi(t), \ \forall t \in \mathbb{R}, \ \forall j \in J.$$ \tag{34}

If $q$ is a linear combination of continuous time weighted functions $\rho_j(t) \cdot q_j(t), \ i = 1, \rho_j \in J$ then the distribution $F_q$ on $\Phi$ is a linear combination, on the same space $\Phi^*$, of the weighted distributions $F_q^j, \ j \in J$, generated by the components $q_j$, and the family of weighting functions $\rho_j, j \in J$

$$q = \sum_{i=1}^{\rho_n} \alpha_i \cdot \rho_i(t) \cdot q_i(t), \Rightarrow$$ \tag{35}

$$F_q(\varphi) = \sum_{i=1}^{\rho_n} \alpha_i \cdot F_q^i(\varphi)$$ \tag{36}

$$F_q^j(\varphi) = \int_R \rho_j(t) \cdot q(t) \cdot \varphi(t) \, dt = F_q^j(\rho_j \cdot \varphi), \ \forall \varphi \in \Phi,$$ \tag{37}

where,

$$F_q^j(\varphi) = \int_R q_j(t) \cdot \varphi(t) \, dt, \ \forall \varphi \in \Phi.$$ \tag{38}

The notion of distribution k-order derivative, $k = 0 \leq n$ (Kecs et al. 1975), is now directly extended to weighted distributions.

If $F_q^\rho \in \Phi^*$, $\rho^j$ of (30), then its $k$-order derivative, $k \leq n$, is a new distribution, $F_q^\rho_{\rho_j}$ uniquely defined by the relations,

$$F_q^{\rho_{j}}(\varphi) = (-1)^{j} \cdot F_q^j([\rho_j \cdot \varphi^{(i)}) \in \Phi^*$$ \tag{39}

where

$$[\rho_j \cdot \varphi^{(i)}] = \frac{d^i[\rho_j(t) \cdot \varphi(t)]}{dt^i} = \sum_{i=1}^{k=0} C_{k}^i \cdot \rho_j^{(i)} \cdot \varphi^{(k-i)}$$ \tag{40}

is the $k$-order ($k = 0 \leq n$) time derivative of the testing function $\varphi_{\rho_j} = \rho_j \cdot \varphi \in \Phi^*$, and

$$\varphi^{(j)} : \mathbb{R} \rightarrow \mathbb{R}, t \rightarrow \varphi^{(j)}(t) = \frac{d^j \varphi(t)}{dt^j}$$ \tag{41}

$$\rho_j^{(j)} : \mathbb{R} \rightarrow \mathbb{R}, t \rightarrow \rho_j^{(j)}(t) = \frac{d^j \rho_j(t)}{dt^j}.$$ \tag{42}

Taking into account (40), (39) can be expressed as

$$F_q^{\rho_{j}}(\varphi) = (-1)^{j} \cdot \sum_{i=1}^{k=0} C_{k}^i \cdot F_q^{\rho_{j}}(\varphi^{(k-i)}), \varphi \in \Phi^*$$ \tag{43}

where,

$$\varphi \rightarrow F_q^{\rho_{j}}(\varphi) = \int_R q_j(t) \cdot \rho_j^{(j)}(t) \cdot \varphi(t) \, dt \in \mathbb{R}$$ \tag{44}
is a weighted distribution generated by function \(q\). When \(q \in C^4(\mathbb{R})\), then

\[
F^{\rho_{\phi(t)}}_{q}(\phi) = F^{\rho_{\phi(t)}}_{q}(\phi) = \int_{\mathbb{R}} q^{(k)}(t) \cdot \rho_j(t) \cdot \phi(t) \, dt
\]

that means the k-order derivative of a weighted distribution generated by a function \(q \in C^4(\mathbb{R})\) equals to the weighted distribution generated by \(q^{(k)}\), the k-order time derivative of the function \(q\),

\[
q^{(k)} : \mathbb{R} \to \mathbb{R}, t \to q^{(k)}(t) = d^k q(t)/dt^k.
\]

If \(q \in C^4(\mathbb{R})\), from (39), (45) one can write,

\[
F^{\rho_{\phi(t)}}_{q}(\phi) = \int_{\mathbb{R}} q^{(k)}(t) \cdot \rho_j(t) \cdot \phi(t) \, dt = (-1)^k \sum_{j=1}^{k} C_k^j \cdot \left[ q(t) \cdot \rho_j^{(k)}(t) \cdot \phi(t) \right] \, dt, \forall \phi \in \Phi_n
\]

For example,

\[
F^{\rho_{\phi(t)}}_{q}(\phi) = (-1)^k \left[ F_{q}^{\rho_{\phi(t)}}(\phi^{(k)}) + k \cdot F_{q}^{\rho_{\phi(t)}}(\phi^{(k-1)}) \right], \forall \phi \in \Phi_n
\]

4. PROBLEM STATEMENT OF CONTINUOUS TIME VARIANT SYSTEM IDENTIFICATION

Let us consider a dynamical continuous time variant system with \(n_i\) inputs,

\[
u : \mathbb{R} \to \mathbb{R}^n, t \to \nu(t), \ u \in \Omega
\]

and \(n_o\) outputs,

\[
y : \mathbb{R} \to \mathbb{R}^n, t \to \nu(t), \ y \in \Gamma
\]

where \(\Omega\) represents the set of admissible inputs and \(\Gamma\) is the set of possible outputs. It can be expressed by a differential operator,

\[
q_{\theta_{(u,y)}}(t) = \bar{Q}(u,y,t,\theta)
\]

whose expression depends on a vector of parameters

\[
\theta = [\theta \ ... \ \theta_k] \cdot
\]

A triple \((u^*, y^*, \theta^* )\) is a realization of the model if the function

\[
q_{\theta_{(u^*, y^* , \theta^*)}} = \bar{Q}(u^*, y^*, t, \theta^*)
\]

is the zero function, \(q_{\theta_{(u^*, y^* , \theta^*)}} = 0\) that means,

\[
q_{\theta_{(u^*, y^* , \theta^*)}}(t) = \bar{Q}(u^*(t), y^*(t), \theta^* ) = 0, \ \forall t \in \mathbb{R}.
\]

The value \(\theta = \theta^*\) is consistent with the model (53) if and only if the two following conditions are accomplished:

1. Covering condition

\[
q_{\theta_{(u,y)}}(t) = \bar{Q}(u(t), y(t), \theta^*) = 0, \ \forall t \in \mathbb{R}, \forall (u(t), y(t)) \in \Omega \times \Gamma
\]

2. Uniqueness condition

\[
q_{\theta_{(u,y)}}(t) = 0, \ \forall t \in \mathbb{R}, \forall (u(t), y(t)) \in \Omega \times \Gamma \Rightarrow \theta = \theta^*
\]

A special case is the model (51) expressing a linear relation in the parameters

\[
q_{\theta_{(u,y)}} = \bar{Q}(u, y, t, \theta) = \sum_{i=1}^{p} w_i \cdot \theta - v = w^\top \cdot \theta - v,
\]

where \(w_i\) and \(v\) represent a sum of the time weighted derivatives of some known, possible nonlinear, functions \(\psi^j_i, \psi^j_o\), with respect to the input and output variables,

\[
w_i = \sum_{j=1}^{p} \rho_j(t) \cdot [\psi^j_i(u(t), y(t))]^{(i)}, i = 1: p,
\]

\[
v = \sum_{j=1}^{p} \rho_j(t) \cdot [\psi^j_o(u(t), y(t))]^{(i)}.
\]

The weighting functions \(\rho_j(t), i = 0: p, j = 1: p\) are given time functions, depending how the time variation of the coefficients is parameterized. In this paper, considering parameterization by polynomials, these functions are of the form

\[
\rho_j(t) = t^m, i = 0: p, j = 1: p_i
\]

Parameters \(p_i, n_i, p_o, n_o, m_i\) are given integer numbers. The identification problem, into condition (57), has a unique solution. At any time instant \(t\), for a measured input-output pair of functions \((u, y)\) the value of the function \(q_{\theta_{(u,y)}}(t)\) is a real vector

\[
q_{\theta_{(u,y)}}(t) = \sum_{i=1}^{p} w_i(t) \cdot \theta - v(t) = w^\top \cdot \theta - v(t)
\]

where

\[
w^\top(t) = [w_1(t),..., w_p(t),..., w_p(t)],
\]

\[
v(t) = \sum_{j=1}^{p} \rho_j(t) \cdot [\psi^j_o(u(t), y(t))]^{(i)}, i = 1: p
\]

Practically, it is possible to record the functions \((u, y)\) in the time interval \(T \in \mathbb{R}\) only, called observation time interval or just time window. The restriction of the functions \((u, y)\) to the time interval \(T\) is denoted by \((u_T, y_T)\) respectively. If no confusion would appear, then we may drop the subscript \(T\).

As in (Marin et al., 2005), an identification problem means to determine the parameter \(\theta = \theta^*\), given the priori information on the model structure \(\bar{Q}\), (51), and the observed input-output pair \((u_T, y_T)\),

\[
\hat{\theta} = \theta(u_T, y_T, Q)
\]

in a such a way that,

\[
q_{\theta_{(u,y)}}(t) = 0, \forall t \in \mathbb{R}
\]

This condition involves,

\[
q_{\theta_{(u,y)}}(t) = 0, \forall t \in \mathbb{R}, \forall (u(t), y(t)) \in \Omega \times \Gamma
\]

for any input-output pair \((u, y)\) observed to that system. As the unknown parameter \(\theta\) has a finite number \(p\) of components, then it will be enough to choose (utilize) a finite number \(N\) of time instants,

\[
T = [t_1, t_2, ..., t_N], i = 1: N
\]

based on which to create an algebraic equation. In the specific case of (48), this is a linear system

\[
W \cdot \theta = v
\]
where $W$ is a $N \times p$ matrix of real numbers,
\[
W = [w(t_1)^T, \ldots, w(t_p)^T]\,.
\] (68)
whose i-th row ($i = 1, N$) is,
\[
w(t_i)^T = [w_i(t_1), \ldots, w_i(t_p)] \quad (69)
\]
The symbol $v$ denotes a $N$ column real vector,
\[
v(t) = [v(t_1), \ldots, v(t_p)]^T \,.
\] (70)
Let us denote $r = \text{rank}(W)$. If $r = p$, then a unique solution is obtained,
\[
\theta = (W^T \cdot W)^{-1} \cdot W^T \cdot v = \theta^* \quad (71)
\]
The equation (71) is of no practical interest because it is not recommended to measure (or to estimate) the derivatives of signals, mainly when they are noise contaminated.

5. DISTRIBUTION APPROACH OF TIME VARIANT SYSTEM IDENTIFICATION

Now let us consider known the set of continuous time scalar functions (62), (63)
\[
w_i(t) = \sum_{j=1}^{p} \rho_j^i(t) \cdot [y_j'(u(t), y(t))]^{(s_j^i)} = \quad (72)
\]
\[
v(t) = \sum_{j=1}^{p} \rho_j^0(t) \cdot [y_j'(u(t), y(t))]^{(s_j^0)} = \quad (73)
\]
The function $w_i(t)$ from (72) generates a distribution
\[
F_n(\phi) = \sum_{j=1}^{p} F_{n,i}^{(j)}(\phi) \quad (74)
\]
where
\[
F_{n,i}^{(j)}(\phi) = F_{n,i}^{(j)}(\phi^*) = \quad (75)
\]
\[
= (-1)^{s_j^i} \cdot \sum_{k=1}^{c_{s_j^i}} \cdot C_{s_j^i}^{k,j} \cdot F_{n,i}^{(j)}(\phi^{(s_j^i-k)}) \forall \phi \in \Phi_n \quad (76)
\]
\[
F_{n,i}^{(j)}(\phi^{(s_j^i-k)}) = \left[ y_j'(u(t)) \right] \cdot [\rho_j^i(t)]^{(s_j^i-k)} \cdot \phi^{(s_j^i-k)}(t) \cdot dt \quad (77)
\]
The result constitutes the row vector,
\[
F_n(\phi) = [F_{n,i}(\phi), \ldots, F_{n,i}(\phi)]^T \in \mathbb{R}^p \quad (78)
\]
Also, the function $v(t)$ from (73) generates a distribution
\[
F_v(\phi) = F_{v,i}^{(j)}(\phi) \quad (79)
\]
where
\[
F_{v,i}^{(j)}(\phi) = F_{v,i}^{(j)}(\phi^*) \quad (80)
\]
\[
= (-1)^{s_j^0} \cdot \sum_{k=1}^{c_{s_j^0}} \cdot C_{s_j^0}^{k,j} \cdot F_{v,i}^{(j)}(\phi^{(s_j^0-k)}) \forall \phi \in \Phi_n \quad (81)
\]
\[
F_{v,i}^{(j)}(\phi^{(s_j^0-k)}) = \left[ y_j(u(t)) \right] \cdot [\rho_j^0(t)]^{(s_j^0-k)} \cdot \phi^{(s_j^0-k)}(t) \cdot dt \quad (82)
\]
\[
F^{(j)}_{\phi | \phi^{(s_j^i-k)}}(\phi^{(s_j^i-k)}) = \int_{\mathbb{R}} \left[ y_j'(t) \right] \cdot [\rho_j^i(t)]^{(s_j^i-k)} \cdot \phi^{(s_j^i-k)}(t) \cdot dt \quad (83)
\]
Into this conditions, any input-output pair $(u, y)$ observed from the system (51) is described by a pair of regular distribution $(F_u, F_y)$ for any $\phi \in \Phi_n$.

In such a way, the problem of identification regarding the parameters of the real system (51) can be represented by distributions. For example, the regular distribution generated by the continuous function $q_{\theta(x,y)}$ from (51), into the specific case of (61) is related to the parameter vector $\theta$ as
\[
F_{q_{\theta(x,y)}}(\phi) = \sum_{i=1}^{p} F_{n,i}(\phi) \cdot \theta_i - F_v(\phi) = \quad (84)
\]
If a triple $(u^*, y^*, \theta^*)$ is a realization of the model (51), then the identity (85) takes place,
\[
F_{q_{\theta(x,y)}}(\phi) = F_{q_{\theta(x,y)}}(\theta^*) = 0, \forall \phi \in \Phi_n \quad (85)
\]
and vice versa, if an input-output pair $(u^*, y^*)$ of the family of models (51), with unknown parameter $\theta$, generates a distribution
\[
F_{q_{\theta(x,y)}}(\phi) = F_{q_{\theta(x,y)}}(\phi) - \sum_{i=1}^{p} F_{n,i}(\phi) \cdot \theta_i - F_v(\phi) = \quad (86)
\]
which satisfies
\[
F_{q_{\theta(x,y)}}(\phi) = F_{q_{\theta(x,y)}}(\phi^*) = 0, \forall \phi \in \Phi_n \quad (87)
\]
and then $\theta = \theta^*$.

As $\theta$ has $p$ components it is enough a chose (utilize) a finite number $N \geq p$ of fundamental functions $\phi_i, i = 1 : N$ and to build an algebraic equation,
\[
F_u \cdot \theta = F_v \quad (88)
\]
where $F_u$ is an $(N \times p)$ matrix of real numbers
\[
F_u = [F_u^0(\phi_1), \ldots, F_u^0(\phi_N), \ldots, F_u^p(\phi_N)]^T \quad (89)
\]
where i-th row $F^i_u(\phi)$ is given by (78).

The symbol $F$ denotes an $N$-column real vector built from (80)-(83),
\[
F = [F(\phi_1), \ldots, F(\phi_p), \ldots, F(\phi_N)]^T \quad (90)
\]
When only the restriction $(u, y)$ of the pair $(u, y)$ on the time interval $T$, is available, then one must choose $\phi_i$ such that
\[
\text{supp}(\phi_i) \subset T \subset \mathbb{R} \, i = 1 : N \quad (91)
\]
If $r = \text{rank}(F_u) = p$, then a unique solution is obtained.
\[
\theta = (F_u^T \cdot F_u)^{-1} F_u^T \cdot F_v = \theta^* \quad (92)
\]

6. THE HIERARCHICAL STRUCTURE OF IDENTIFICATION AND ESTIMATION EQUATIONS

The procedure of hierarchical structure identification developed in (Marin et al., 2005) is retaken here considering time variant coefficients (10) and (11). To obtain linear equations in unknown parameters, the identification problem is split in several simpler
interlinked identification problems called identification layers. Based on the specific structure of this system, it is possible to group the state equations, in such way to determine five interconnected identification problems of the type (92), labeled Layer__, __, __, __, __. They are organized in a hierarchical structure. First, in Layer_a, some state equations are utilized to obtain a set of linear equations in some parameters. The results of this first stage of identification are utilized for expressing other parameters by linear equations in Layer_b. This process is repeated in the other layers until all parameters are identified. For each identification layer, the same type of procedures and numerical algorithms are applied.

Layer_a: Identification of [θ̃1, θ̃0], θ̃(t) = θ̃1 + t + θ̃0

Substituting expression φ from (16) into (18) we obtain, the Layer_a model (51) characterized by

\[ \theta^* = [\theta^* \theta^0] = [\theta\ \theta_0], \ \ p^* = 2 \]
\[ w^*(t) = [\xi^0 \cdot \xi^2 \cdot \theta^0 \cdot \theta_0]^0 \]
\[ w^0(t) = [\xi^0 \cdot [u_0 \cdot u_1 - \xi^0]^0 - [u_0 \cdot u_1]^0 \]
\[ v^*(t) = [-\frac{1}{2} \xi^0 \cdot \xi^2 \cdot [u_0 \cdot u_1 \cdot \xi^0][u_0 \cdot u_1 \cdot \xi^0]^0 \]
\[ F^*(\phi) = \left[ [\xi^0 \cdot \xi^2 \cdot \theta^0 \cdot \theta_0 \cdot \phi^0 \cdot \phi_0]^0 \right] \]
\[ F^0_w(\phi) = \left[ [\xi^0 \cdot \xi^2 \cdot \theta^0 \cdot \theta_0 \cdot \phi^0 \cdot \phi_0]^0 \right] + \int_{\mathbb{R}} [u_0(t) \cdot u_1(t) \cdot \phi^0(t) \cdot dt + \int_{\mathbb{R}} \left[ [u_0(t) \cdot u_1(t) \cdot \xi^0]^0 + [u_0 \cdot u_1 \cdot \xi^0]^0 \right] \]
\[ F^b(\phi) = [F^b_w(\phi) F^b_w(\phi) F^b_w(\phi)] \]
\[ \theta^* = \left[ F^b_w(\phi) F^b_w(\phi) F^b_w(\phi) \right] \]
\[ \hat{\theta}_1 = \hat{\theta}_2^*; \ \hat{\theta}_0 = \hat{\theta}_0^* \]
\[ \hat{\theta}(t) = \hat{\theta}_1 + t + \hat{\theta}_0 \]

Layer_b: Identification of θ̃1, θ̃0, θ̃

Considering known θ̃1, θ̃0, θ̃ from the Layer_b the estimated expression φ̂, of the rational φ̂, is

\[ \phî = \hat{\theta}_1 \cdot \hat{\xi}_1 \cdot \hat{\xi}_2 \]

Substituting expression φ from (19) and (101) instead of φ into (21) we obtain,

\[ \hat{\xi}_1 = \left[ [\xi^0 \cdot \xi^2 \cdot \theta^0 \cdot \theta_0 \cdot \phi^0 \cdot \phi_0]^0 \right] - [u_0 \cdot \xi^0]^0 \]

which determines the Layer_c model (51) characterized by

\[ \theta^* = [\theta^* \theta^0] = [\theta\ \theta_0], \ \ p^* = 3 \]
\[ w^*(t) = [\xi^0 \cdot \xi^2 \cdot \theta^0 \cdot \theta_0]^0 \]
\[ w^0(t) = [\xi^0 \cdot [u_0 \cdot u_1 - \xi^0]^0 \]
\[ v^*(t) = [-\frac{1}{2} \xi^0 \cdot \xi^2 \cdot [u_0 \cdot u_1 \cdot \xi^0][u_0 \cdot u_1 \cdot \xi^0]^0 \]
\[ F^*(\phi) = \left[ [\xi^0 \cdot \xi^2 \cdot \theta^0 \cdot \theta_0 \cdot \phi^0 \cdot \phi_0]^0 \right] + \int_{\mathbb{R}} [u_0(t) \cdot u_1(t) \cdot \phi^0(t) \cdot dt + \int_{\mathbb{R}} \left[ [u_0(t) \cdot u_1(t) \cdot \xi^0]^0 + [u_0 \cdot u_1 \cdot \xi^0]^0 \right] \]
\[ F^b(\phi) = [F^b_w(\phi) F^b_w(\phi) F^b_w(\phi)] \]
\[ \theta^* = \left[ F^b_w(\phi) F^b_w(\phi) F^b_w(\phi) \right] \]
\[ \hat{\theta}_1 = \hat{\theta}_2^*; \ \hat{\theta}_0 = \hat{\theta}_0^* \]
\[ \hat{\theta}(t) = \hat{\theta}_1 \cdot \hat{\xi}_1 \cdot \hat{\xi}_2 \]

Layer_c: Identification of θ̃1, θ̃0, θ̃

Considering known θ̃1, θ̃0, θ̃ from the Layer_b the estimated expression φ̂, of the rational φ̂, is

\[ \phî = \hat{\theta}_1 \cdot \hat{\xi}_1 \cdot \hat{\xi}_2 \]

Substituting expression φ̂ from (19) and (101) instead of φ̂ into (21) we obtain,

\[ \hat{\xi}_1 = \left[ [\xi^0 \cdot \xi^2 \cdot \theta^0 \cdot \theta_0 \cdot \phi^0 \cdot \phi_0]^0 \right] - [u_0 \cdot \xi^0]^0 \]

which determines the Layer_c model (51) characterized by

\[ \theta^* = [\theta^* \theta^0] = [\theta\ \theta_0], \ \ p^* = 3 \]
\[ w^*(t) = [\xi^0 \cdot \xi^2 \cdot \theta^0 \cdot \theta_0]^0 \]
\[ w^0(t) = [\xi^0 \cdot [u_0 \cdot u_1 - \xi^0]^0 \]
\[ v^*(t) = [-\frac{1}{2} \xi^0 \cdot \xi^2 \cdot [u_0 \cdot u_1 \cdot \xi^0][u_0 \cdot u_1 \cdot \xi^0]^0 \]
\[ F^*(\phi) = \left[ [\xi^0 \cdot \xi^2 \cdot \theta^0 \cdot \theta_0 \cdot \phi^0 \cdot \phi_0]^0 \right] + \int_{\mathbb{R}} [u_0(t) \cdot u_1(t) \cdot \phi^0(t) \cdot dt + \int_{\mathbb{R}} \left[ [u_0(t) \cdot u_1(t) \cdot \xi^0]^0 + [u_0 \cdot u_1 \cdot \xi^0]^0 \right] \]
\[ F^b(\phi) = [F^b_w(\phi) F^b_w(\phi) F^b_w(\phi)] \]
\[ \theta^* = \left[ F^b_w(\phi) F^b_w(\phi) F^b_w(\phi) \right] \]
\[ \hat{\theta}_1 = \hat{\theta}_2^*; \ \hat{\theta}_0 = \hat{\theta}_0^* \]
\[ \hat{\theta}(t) = \hat{\theta}_1 \cdot \hat{\xi}_1 \cdot \hat{\xi}_2 \]
Considering known \( \theta_0 = \theta_1 \); \( \theta_1 = \hat{\theta}_1 \); \( \theta_2 = \hat{\theta}_2 \), from the Layer d identification, the estimated expression \( \hat{\theta}_1 \), of the nonlinear function \( \phi_1 \), is
\[
\hat{\phi}_1 = \frac{\xi_1 \cdot \xi_2}{\theta_1 + \xi_1 + \xi_2} \cdot \hat{\theta}_1 (t) \cdot \hat{\xi}_1 \cdot \hat{\xi}_2 + u_1 \cdot \xi_1 \cdot \xi_2 - u_3 \cdot \xi_3.
\]
The Layer d model (51) is now,
\[
\theta'[t] = [\xi_1 \cdot \xi_2 \cdot \theta_1 \cdot \theta_2 \cdot \theta_3][t] \quad p' = 3
\]
\[
\nu_i'[t] = [\xi_1 \cdot \xi_2 \cdot \nu_1 \cdot \nu_2 \cdot \nu_3][t] \quad u_i'[t] = [\xi_1 \cdot \xi_2 \cdot \nu_1 \cdot \nu_2 \cdot \nu_3][t]
\]
\[
F'_{\nu}(\nu)[t] = \left[\xi_1 \cdot \xi_2 \cdot \nu_1 \cdot \nu_2 \cdot \nu_3\right][t] \cdot \nu_0(t) \cdot dt
\]
\[
F'_{\nu}(\nu)[t] = \left[\xi_1 \cdot \xi_2 \cdot \nu_1 \cdot \nu_2 \cdot \nu_3\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt
\]
Also,
\[
F'_{\nu}(\nu)[t] = \left[\xi_1 \cdot \xi_2 \cdot \nu_1 \cdot \nu_2 \cdot \nu_3\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt
\]
\[
F'_{\nu}(\nu)[t] = \left[\xi_1 \cdot \xi_2 \cdot \nu_1 \cdot \nu_2 \cdot \nu_3\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt
\]
\[
\hat{\theta}_2 = \frac{\nu_2(t)}{\theta_1(t) \cdot \xi_1(t) + \xi_2(t) + \theta_0 \cdot \xi_2(t)} \quad \hat{\theta}_3 = \frac{\nu_3(t)}{\theta_1(t) \cdot \xi_1(t) + \xi_2(t) + \theta_0 \cdot \xi_2(t)}
\]
were expressed instead of \( \phi_2 \) into (22),
\[
\hat{\phi}_2 = -u_1 \cdot \xi_1 + \theta_4 \cdot \phi_2 \cdot u_2
\]
which determines the Layer e model (40)
\[
\theta''[t] = [\xi_1 \cdot \xi_2 \cdot \theta_1 \cdot \theta_2 \cdot \theta_3][t] \quad p'' = 1
\]
\[
\nu_i''[t] = [\xi_1 \cdot \xi_2 \cdot \nu_1 \cdot \nu_2 \cdot \nu_3][t] \quad u_i''[t] = [\xi_1 \cdot \xi_2 \cdot \nu_1 \cdot \nu_2 \cdot \nu_3][t]
\]
\[
F''_{\nu}(\nu)[t] = \left[\xi_1 \cdot \xi_2 \cdot \nu_1 \cdot \nu_2 \cdot \nu_3\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt
\]
\[
F''_{\nu}(\nu)[t] = \left[\xi_1 \cdot \xi_2 \cdot \nu_1 \cdot \nu_2 \cdot \nu_3\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt + \left[\theta_1(t) \cdot \theta_2(t) \cdot \theta_3(t)\right][t] \cdot \nu_0(t) \cdot dt
\]
\[
\hat{\theta}_2 = \frac{\nu_2(t)}{\theta_1(t) \cdot \xi_1(t) + \xi_2(t) + \theta_0 \cdot \xi_2(t)} \quad \hat{\theta}_3 = \frac{\nu_3(t)}{\theta_1(t) \cdot \xi_1(t) + \xi_2(t) + \theta_0 \cdot \xi_2(t)}
\]
7. EXPERIMENTAL RESULTS
All Matlab programs developed in (Marin et al., 2005) for time invariant systems have been extended to the time variant systems according to the above relations.
Because of limited space, here we include practical results only for a first order nonlinear time variant system characterized by rational dependence on parameters,
\[
y(t) = \theta_1 \cdot y(t) + \frac{\theta_2(t) \cdot u(t)}{\theta_3 + y(t)}
\]
where
\[
\theta(t) = [\theta_1 \cdot \theta_2 \cdot \theta_3, \theta_0] \quad \theta_0 = \theta_1 \cdot t + \theta_0
\]
The vector of unknown parameters to be identified is
\[
\theta = [\theta_1 \cdot \theta_2 \cdot \theta_3, \theta_0]
\]
This simple model corresponds somehow to the Layer a presented above. It has been utilized testing functions \( \phi(t) \), characterized by a bounded support
\[
T = [t_1, t_2], t_1 < t_2 \quad \text{ accomplishing the conditions}
\]
\[
\phi(t) = 0, \forall t \in (-\infty, t_1) \cup (t_2, +\infty)
\]
\( \varphi(t) = \alpha \cdot \beta(t_s, t_b) \cdot \Psi(t, t_s, t_b) \)

where \( \Psi(t, t_s, t_b) = \sin[\pi \cdot (t - t_b)/(t_s - t_a)] \)

\( \alpha \) is a scaling factor and

\[ \beta(t_s, t_b) = \frac{1}{t_s - t_a} \int_{t_a}^{t_b} \Psi(t, t_s, t_b), \forall t_s, t_b \]

to assure a normalized area.

Figure 1 shows the time variation of unknown coefficients. Two of them are constant \( \theta_1 = a_1 = -0.9 \); \( \theta_2 = a_2 = 2 \), and the third is a sequence of four lines

\[ \theta(t) = a_i(t) = \theta_{1i} \cdot t + \theta_{2i} \]

Fig. 1. Time variation of unknown parameters

The measured input-output variables, for the free noise case, are depicted in Figure 2.

Fig. 2. Time variation of measured input-output variables.

For the first three slopes, the measured variables and identification results are presented as follows:

Fig. 3. Time variation of measured input-output variables for the first slope.

Table 1. Results for the first slope

<table>
<thead>
<tr>
<th></th>
<th>Real</th>
<th>Identified</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_2 )</td>
<td>2.0</td>
<td>2.000000002893786</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>-0.9</td>
<td>-0.90000000953150</td>
</tr>
<tr>
<td>( \theta_{31} )</td>
<td>0.0</td>
<td>0.0000000015214</td>
</tr>
<tr>
<td>( \theta_{30} )</td>
<td>1.0</td>
<td>1.000000001485449</td>
</tr>
</tbody>
</table>

Considering measured output contaminated by noise for the input-output pair from Fig. 6 and Fig. 7, the identification results for the second slope and Fig. 7, the identification results for the second slope are presented in Table 4.

Table 2. Results for the second slope

<table>
<thead>
<tr>
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<th>Real</th>
<th>Identified</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_2 )</td>
<td>2.0</td>
<td>2.000000007054610</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>-0.9</td>
<td>-0.90000001125488</td>
</tr>
<tr>
<td>( \theta_{31} )</td>
<td>-0.144</td>
<td>-0.14400000531677</td>
</tr>
<tr>
<td>( \theta_{30} )</td>
<td>2.8</td>
<td>2.80000010349192</td>
</tr>
</tbody>
</table>

Table 3. Results for the third slope

<table>
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<th>Identified</th>
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<tbody>
<tr>
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<td>2.0</td>
<td>1.99999950652750</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>-0.9</td>
<td>-0.8999991867011</td>
</tr>
<tr>
<td>( \theta_{31} )</td>
<td>0.08</td>
<td>0.0799998039933</td>
</tr>
<tr>
<td>( \theta_{30} )</td>
<td>-2.8</td>
<td>-2.7999931402201</td>
</tr>
</tbody>
</table>

Fig. 4. Time variation of measured input-output variables for the second slope.

Fig. 5. Time variation of measured input-output variables for the third slope.

Fig. 6. Time variation of measured variables with noise on output.
8. CONCLUSIONS

Through this research has been proved that it is possible to identify all parameters of continuous time nonlinear systems even if they are related to measured variables by rational expressions. This is possible if the identification problem is formulated as a set of interconnected identification problems with linear dependences between parameters and measured variables. The problem of functionals based identification consistency has to be analyzed for a broader class of nonlinear systems.

ACKNOWLEDGMENTS

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REFERENCES


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</tr>
<tr>
<td>$\theta_3$</td>
<td>-0.9</td>
</tr>
<tr>
<td>$\theta_{31}$</td>
<td>-0.144</td>
</tr>
<tr>
<td>$\theta_{50}$</td>
<td>2.8</td>
</tr>
</tbody>
</table>

![Image: Time variant NCTS identification]