THE MATHEMATICAL MODEL OF A WALKING ROBOT WITH THREE FREE JOINTS

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Abstract: This paper presents the mathematical model of a planar robot where the active pair of leg has three free joint. The mathematical model of the robot is determined considering all the points in the xz-plane as being complex numbers. Taking into account a possible symmetrical structure, only the vertical xz-plane evolution is considered. The results can be extended to three-dimensional space.

The approach is best on variable dynamic systems, considering that each leg of the pair can change the state depending of the causality order.

The approach is also best on the mechanical structure of a planar robot presented in (Petrişor, Marin 2003).

It is considered that the first leg has both joints free and the second leg has the second joint free.

A systemic approach of the Variable Causality Dynamic Systems (VCDS) is presented with results which allow an easy numerical implementation.

The model is implemented in MATLAB environment and some evolutions examples are presented.

Key words: walking robots, causal ordering, control algorithms.

1. INTRODUCTION

Behavior of walking robots is characterized by a specific type of movement called legged locomotion, (Thirion, et al., 2001), (Cubero, 2001). Legged locomotion combines continuous time differential systems and logical systems concepts, which allow describing the both fundamental aspects: leg movements and leg coordination. As a result, different types of legged movements are possible as walking gates and climbing movements.

Many control algorithms implemented on the existing walking robots, (WMC 2003), (CWR 2003), are based on "state of the art" technologies to control the movements of articulated limbs and joint actuators.

This paper presents the mathematical model of a planar robot where the active pair of leg has three free joint. The mathematical model of the robot is determined considering all the points in the xz-plane as being complex numbers. Taking into account a possible symmetrical structure, only the vertical xz-plane evolution is considered. The results can be extended to three-dimensional space.

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2. GEOMETRICAL STRUCTURE

Let us consider a planar walking robot (PWR) structure as depicted in Fig.1. , having three normal legs L^i, L^j, L^p and a head equivalent to another leg L_0 containing the robot centre of gravity G placed in its foot. The robot body RB is characterised by two position vectors O^0, O^1 and the leg joining points (hips) denoted R^i, R^j, R^p. The joining point of the head L^0 is the central point O^0, $R^0 = O^0$, so the robot body RB is univocally characterized by the set,

$$RB = \{O^0, O^1, \lambda^i, \lambda^j, \lambda^p, \lambda^0\}, \qquad (1)$$

where $\lambda^0 = 0$. The robot has a rigid body if the three scalars $(\lambda^i, \lambda^j, \lambda^k)$ are constant in time.

The geometrical structure of the PWR is defined by

$$O^1 - O^0 = e^{j\cdot\theta} \qquad (2)$$

$$R^i = O^0 + \lambda^i \cdot e^{j\cdot\theta} \qquad (3)$$

$$R^j = O^0 + \lambda^j \cdot e^{j\cdot\theta} \qquad (4)$$

$$R^p = O^0 + \lambda^p \cdot e^{j\cdot\theta} \qquad (5)$$

$$R^0 = O^0 + \lambda^0 \cdot e^{j\cdot\theta} = O^0 \qquad (6)$$

from which,

$$R^i - R^j = (\lambda^i - \lambda^j)\cdot e^{j\cdot\theta} \qquad (7)$$

$$R^p - R^j = (\lambda^p - \lambda^j)\cdot e^{j\cdot\theta} \qquad (8)$$

$$R^i - R^p = (\lambda^i - \lambda^p)\cdot e^{j\cdot\theta} \qquad (9)$$

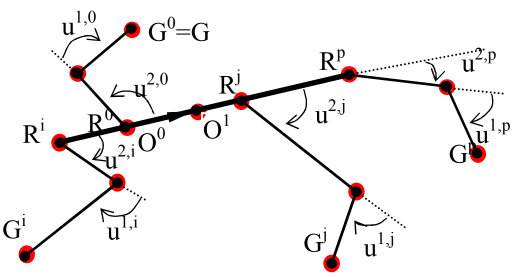


Fig.1. The robot geometrical structure.

The robot position in the vertical plane is defined by the pair of the position vectors O^0, O^1 where $|O^1 - O^0| = 1$, or by the vector O^0 and the scalar θ, the angular direction of the robot body.

Each of the four robot legs L^i, L^j, L^p, L^0, is characterised by an so called Existence Relation $ER(L)$ depending on specific variables as we presented in (Petrişor, Marin 2003).

The mathematical model of this object is a Variable Causality Dynamic Systems VCDS (Marin, 2003) and will be analysed from this point of view.

A pair of legs $\{L_i, L_j\}$ constitutes the so called Active Pair of Legs (APL) if the robot body position is the same irrespective of the feet position of all the other legs different of L_i and L_j. A label is assigned to each possible APL. The APL label is expressed by a variable q called Index of Activity (IA) which can take N_a values, numbers or strings of characters. For example the string of characters, $q = 'ij'$ points out that the pair $\{L_i, L_j\}$ is an APL. Instead of strings of characters the IA can take numerical values as for example,

$$\begin{aligned} q = 12 &\Leftrightarrow q = 'ij' \\ q = 23 &\Leftrightarrow q = 'jp' \\ q = 31 &\Leftrightarrow q = 'pi' \end{aligned} \tag{10}$$

All the other legs that at a time instant do not belong to APL are called Passive Legs (PL).

The leg in APL, having a free joining point (FJP) is called slave leg the opposite of the motor (or master) leg whose both joining points are external controlled (EC).

3. CAUSALITY ORDERING OF AN APL WITH THREE FREE JOINTS

Let be APL=$\{L_i, L_j\}$ that means $q = 'ij' \Leftrightarrow q = 12$.

Changing the indices, the below relations are available for any APL.

The kinematics structure of this APL can be followed in Fig.1.

In this structure only one angle is external controlled (EC) so three joints are free. Considering also the pair L_i, L_j as APL we denote this by

$$q = 'ij', s = \left[\text{motor00, motor01, } s^p\right] \tag{11}$$

or

$$q = 'ij', s = \left[\text{motor00, motor02, } s^p\right]. \tag{12}$$

In this paper it is considering only the causality ordering:

s=[motor00, motor 01, s^p]

In this causality ordering the angle $u^{1,j}$ is EC and the angle $u^{2,j}$ is free.

The block diagram of this causality structure, considering $u^{1,j}$ to be EC, is represented as a connection of oriented subsystems, is illustrated in Fig.2.

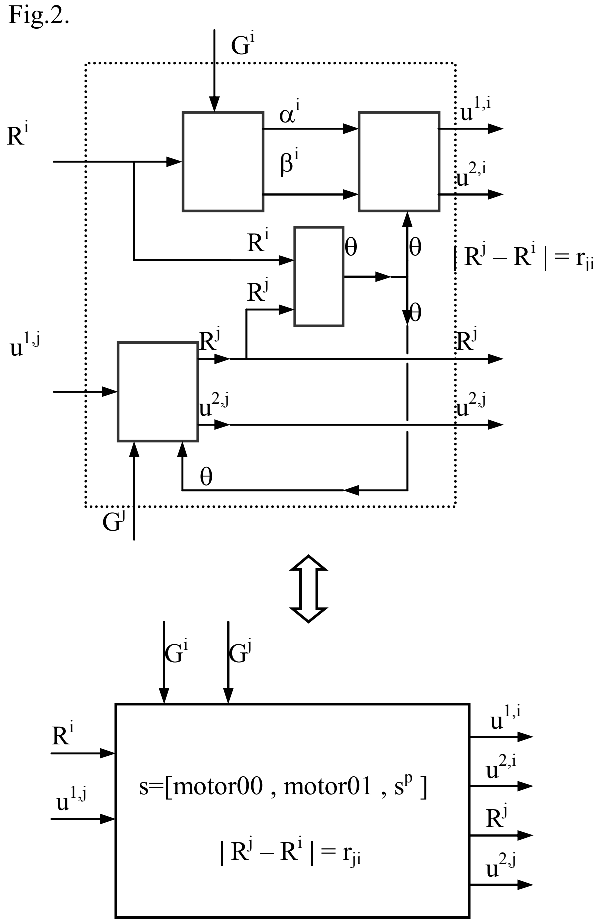


Fig.2. Block diagram of a causality structure with three free angles.

In this structure, the position vectors R^i, G^i, G^j can be external controlled. The vector R^j depends on R^i, G^j and satisfies $\left|R^j - R^i\right| = r_{ji}$. Now, the vector G^i does not affect R^j. The output variables are,

$$R^j = H_j\left(R^i, G^j, u^{1,j}\right) \tag{13}$$

$$u^{1,i} = F_{1i}\left(R^i, G^i, G^j, u^{1,j}\right) \tag{14}$$

$$u^{2,i} = F_{2i}\left(R^i, G^i, G^j, u^{1,j}\right) \tag{15}$$

$$u^{2,j} = F_{2j}\left(R^i, G^i, G^j, u^{1,j}\right) \tag{16}$$

It can be observed that the α^i and β^i depend on R^i and G^i only.

$$\theta = F_{2j}^{\theta}(R^i, G^j, u^{1,j}, s^j) \tag{17}$$

$$\alpha^i = f_\alpha\left(R^i, G^i\right) \tag{18}$$

$$\beta^i = f_\beta\left(R^i, G^i\right) \tag{19}$$

but the command angles $u^{1,i}$, $u^{2,i}$ are referred with respect to the robot body so they depend on the position angle θ so they depend on G^j and $u^{1,j}$.

The relations (17), (18) represent the mathematical model solution of an isolate leg with two free joints.
So, for the leg L_i we have

$$[\alpha^i,\beta^i,s^i]=f_{\alpha\beta}(R^i,G^i,\hat{s}^i,a^i,b^i) \quad (20)$$

The output variable R^j, θ, $u^{2,j}$ are calculated from the existence relations

$$R^j = G^j - e^{j\theta}\cdot AB^j \quad (21)$$

$$AB^j = e^{ju^{2,j}}\cdot[b^j + a^j\cdot e^{j\cdot u^{1,j}}] \quad (22)$$

of the leg L_j and the kinematic restriction

$$R^j - R^i = (\lambda^j-\lambda^i)\cdot e^{j\cdot\theta} \quad (23)$$

where

$$\left|\lambda^j-\lambda^i\right| = r_{ij} = \text{constant} \quad (24)$$

The input-state-output relations of the leg L_j in causality structure motor01 are illustrated in Fig3.

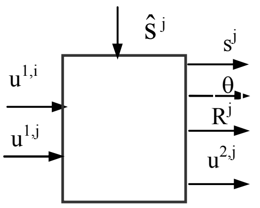


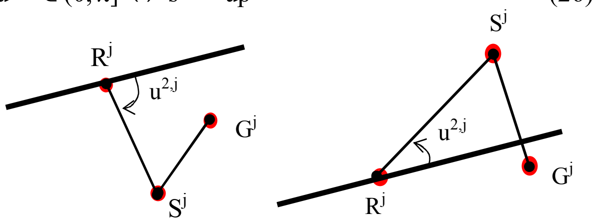
Fig.3. The input-state-output relations of the leg L_j.

For example, the same pair of values (R^j, G^j) can be realized in two ways:

$$u^{2,j}\in[-\pi,0] \Leftrightarrow \hat{s}^j = \text{'down'}$$
(25)

and

$$u^{2,j}\in(0,\pi] \Leftrightarrow \hat{s}^j = \text{'up'} \quad (26)$$



$u^{2,j}\in[-\pi,0] \Leftrightarrow \hat{s}^j = \text{'down'}$ $\quad u^{2,j}\in(0,\pi] \Leftrightarrow \hat{s}^j = \text{'up'}$

Fig.4. The achievement of the values (R^j, G^j).

The equations (21), (23) represent two complex equations, so 4 scalars equations from which are calculated the outputs θ, R^j, $u^{2,j}$ that means two scalars and a complex number.
Replacing for $\lambda^j \neq \lambda^i$

$$e^{j\theta} = \frac{R^j - R^i}{\lambda^j - \lambda^i} \quad (27)$$

from (23), in (21) are obtained

$$R^j = \frac{(\lambda^j-\lambda^i)\cdot G^j + AB^j\cdot R^i}{\lambda^j-\lambda^i+AB^j} = f(u^{1,j},u^{2,j},R^i,G^j) \quad (28)$$

from which on calculate

$$R^j - R^i = (\lambda^j-\lambda^i)\cdot\frac{G^j-R^i}{\lambda^j-\lambda^i+AB^j} \quad (29)$$

By replacing (29) in (27) is obtaine, for $\lambda^j-\lambda^i \neq 0$

$$e^{j\theta} = \frac{G^j-R^i}{\lambda^j-\lambda^i+AB^j} \quad (30)$$

for which

$$\left|e^{j\theta}\right| = \left|\frac{G^j-R^i}{\lambda^j-\lambda^i+AB^j}\right| \quad (31)$$

thus, is obtained the kinematic restriction condition moved to the input,

$$\left|\lambda^j-\lambda^i+AB^j\right| = \left|G^j-R^i\right|. \quad (32)$$

The relation (32), in explicity form becomes,

$$\left|\lambda^j-\lambda^i+e^{ju^{2,j}}\cdot[b^j+a^j\cdot e^{ju^{1,j}}]\right| = \left|G^j-R^i\right| \quad (33)$$

The angle value is obtained from (33)

$$u^{2,j} = \tilde{u}^{2,j} = F_{2j}(u^{1,j},R^i,G^j,\hat{s}^j) \quad (34)$$

The state $\hat{s}^j$ is introduced to restore the input-output univocity.
We denote,

$$W_{2j} = b^j + a^j\cdot e^{ju^{1,j}} \quad (35)$$

where the subscript 2 meaning that angle u^2 (here $u^{2,j}$) is free, so by division with W_{2j},(33) becomes

$$\left|\frac{\lambda^j-\lambda^i}{W_{2j}} + e^{j\cdot u^{2,j}}\right| = \left|\frac{G^j-R^i}{W_{2j}}\right| \quad (36)$$

We denote,

$$Q_{2j} = \frac{\lambda^j-\lambda^i}{W_{2j}} = q_{2j}^x + j\cdot q_{2j}^z = Q_{2j}(u^{1,j}) \quad (37)$$

$$g_{2j} = \left|\frac{G^j-R^i}{W_{2j}}\right| = g_{2j}(u^{1,j},R^i,G^j). \quad (38)$$

It is obtained a polynomial equation, with unknown scalar $u^{2,j}$, presented in the form as

$$\left|Q_{2j} + e^{ju^{2,j}}\right| = g_{2j} \quad (39)$$

It is obtained

$$\left|q_{2j}^x + j\cdot q_{2j}^z + \cos(u^{2,j}) + j\cdot\sin(u^{2,j})\right| = g_{2j} \quad (40)$$

$$(q_{2j}^x)^2 + (q_{2j}^z)^2 + 1 + 2\cdot q_{2j}^x\cdot\cos(u^{2,j}) + 2\cdot q_{2j}^z\cdot(\sin u^{2,j}) = g_{2j}^2$$

a linear equation in sin and cos. (41)
We denote,

$$a_{2j} = 2q_{2j}^x = a_{2j}(u^{1,j}) \quad (42)$$

$$b_{2j} = 2q_{2j}^z = b_{2j}(u^{1,j}) \quad (43)$$

$$c_{2j} = -(q_{2j}^x)^2 - (q_{2j}^z)^2 - 1 + g_{2j} = -\left|Q_{2j}\right|^2 - 1 + g_{2j}^z \quad (44)$$

$$c_{2j} = c_{2j}(u^{1,j},R^i,G^j)$$

From (39), with (42) ÷ (44) is obtained a trigonometric equation

$$a_{2j}\cos(u^{2,j}) + b_{2j}\sin(u^{2,j}) = c_{2j} \quad (45)$$

with the following solutions of the t current moment:

$$u^{2,j} = \tilde{u}^{2,j}(t) = F_{2j}(u^{1,j},R^i,G^j,\hat{s}^j) \quad (46)$$

The case $b_{2j} = 0$ and $a_{2j} \neq 0$

$$\psi_{2j} = \arccos\left(\frac{c_{2j}}{a_{2j}}\right) \in [0, \pi] \text{ , } a_{2j} \neq 0 \quad (47)$$

$$\varphi_{2j} = \begin{cases} \frac{\pi}{2} \text{ if } a_{2j} > 0 \\ -\frac{\pi}{2} \text{ if } a_{2j} > 0 \end{cases} \quad (48)$$

$$\tilde{u}^{2j}(t) = \begin{cases} \psi_{2j} & \text{if} \quad u^{2,j}(t-\varepsilon) > 0 \quad \Leftrightarrow \quad \hat{s}^{j} = \text{'up'} \\ -\psi_{2j} & \text{if} \quad u^{2,j}(t-\varepsilon) \leq 0 \quad \Leftrightarrow \quad \hat{s}^{j} = \text{'down'} \end{cases} \quad (49)$$

The case $b_{2j} = 0$ and $a_{2j} = 0$, is possible only if c_{2j} and we suppose

$$\varphi_{2j} = 0, \ \psi_{2j} = 0, \tilde{u}^{2,j}(t) = 0 \quad (50)$$

The case $b_{2j} \neq 0$

On calculate:

$$\varphi_{2j} = \operatorname{arctg}\frac{a_{2j}}{b_{2j}} \in (-\pi/2 \text{ , } \pi/2) \quad (51)$$

$$\psi_{2j} = \arcsin\left(\frac{|b_{2j}|}{b_{2j}} \cdot \frac{c_{2j}}{\sqrt{(a_{2j})^2 + (b_{2j})^2}}\right) \in [-\pi/2 \text{ , } \pi/2] \quad (52)$$

If $u^{2,j}(t-\varepsilon) \leq 0 \ \Leftrightarrow \ s^{1,j} = \text{'down'}$

$$\tilde{u}^{2,j}(t) = \begin{cases} -\varphi_{2j} + \psi_{2j} & \text{if} \quad -\varphi_{2j} + \psi_{2j} \leq 0 \\ -\varphi_{2j} + \psi_{2j} - \pi & \text{if} \quad -\varphi_{2j} + \psi_{2j} > 0 \end{cases} \quad (53)$$

If $u^{2,j}(t-\varepsilon) > 0 \ \Leftrightarrow \ s^{1,j} = \text{'up'}$

$$\tilde{u}^{2,j}(t) = \begin{cases} -\varphi_{2j} + \psi_{2j} + \pi & \text{if} \quad -\varphi_{2j} + \psi_{2j} \leq 0 \\ -\varphi_{2j} + \psi_{2j} & \text{if} \quad -\varphi_{2j} + \psi_{2j} > 0 \end{cases} \quad (54)$$

The solution (34) of the equation (33) equivalent with (50) is given by one of the expression (49), (50), (53) or (54). Each form, in its validity conditions, respects the kinematic restriction of rigid body (24).

Fig.5 presents a block diagram which illustrate the subordonation of different relations in $\tilde{u}^{2,j}$ angle routine computation

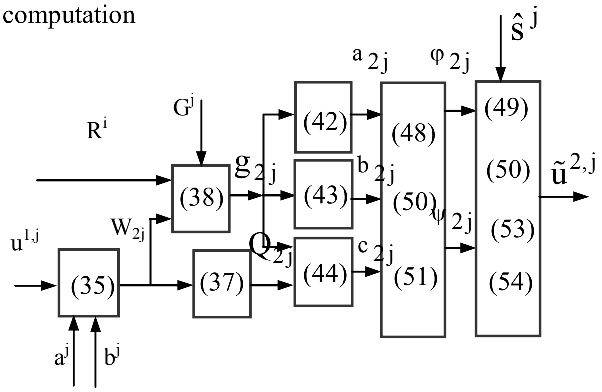


Fig.5. The block diagram.

which is such as (34)

$$\tilde{u}^{2,j} = F_{2j}(u^{1,j}, R^{i}, G^{j}, \hat{s}^{j}) \quad (55)$$

concis illustrated dependence in Fig. 6.

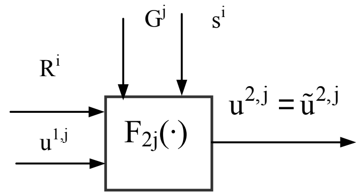


Fig.6. Concis block diagram .

Knowing the angle value $u^{2,j} = \tilde{u}^{2,j}$, on calculate the input variables vector oj the leg L_j

$$\tilde{AB}_2^{j} = e^{j\cdot\tilde{u}^{2,j}} \cdot W_{2j} = e^{j\cdot\tilde{u}^{2,j}} \cdot [b^{j} + a^{j} \cdot e^{j\cdot u^{1,j}}] \quad (56)$$

which assumed the kinematic restriction.

In such of conditions the angle θ and the vector R^{j} are calculated, by replacing (56) in (30) and (28),

$$e^{j\theta} = \frac{G^{j} - R^{i}}{\lambda^{j} - \lambda^{i} + \tilde{AB}_2^{j}} \quad (57)$$

whence

$$\theta = \arg\left[\frac{G^{j} - R^{i}}{\lambda^{j} - \lambda^{i} + \tilde{AB}_2^{j}}\right] = F_{2j}^{\theta}(u^{1,j}, R^{i}, G^{j}, \hat{s}^{j}) \quad (58)$$

$$R^{j} = \frac{(\lambda^{j} - \lambda^{i}) \cdot G^{j} + \tilde{AB}_2^{j} \cdot R^{i}}{\lambda^{j} - \lambda^{i} + \tilde{AB}_2^{j}} = H_{2j}(u^{1,j}, R^{i}, G^{j}, \hat{s}^{j}) \quad (59)$$

Also, the knowledge of the angle θ allows the calculation of the free angles $u^{1,i}$, $u^{2,i}$ of the leg L_i , about which are already known the angles α^{i} , β^{i} estimated by the function $f_{\alpha\beta}(\cdot)$, (20).

The angles $u^{1,i}$, $u^{2,i}$ are

$$u^{1,i} = \alpha^{i} - \beta^{i} \quad (60)$$

$$u^{2,i} = \beta^{i} - \theta - \pi \quad (61)$$

A calculation alternative suppose first assessment of the vector R^{j} by relation (59) and then the angle θ is estimated from the vectors R^{i} şi R^{j} as in relation (62)

$$\theta = \arg\left(\frac{R^{j} - R^{i}}{\lambda^{j} - \lambda^{i}}\right) = \arg\left(\frac{H_{2j}(u^{1,j}, R^{i}, G^{j}, \hat{s}^{j}) - R^{i}}{\lambda^{j} - \lambda^{i}}\right) \quad (62)$$

instead of relation (58).

4. CONCLUSIONS

The mathematical model on VCDS allows the best implementation of the robot behaviour.

Based on relations obtained it was conceived and implemented computer programs for simulation and control.

5. EXAMPLES OF EVOLUTIONS USING MATLAB ENVIRONMENT

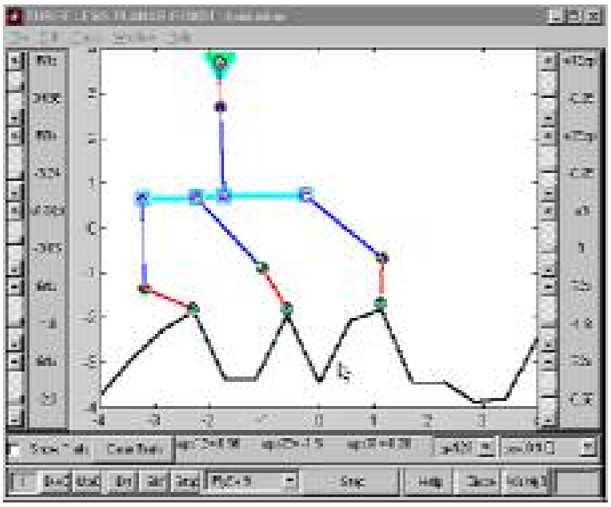


Fig.8. Initial state of the robot .

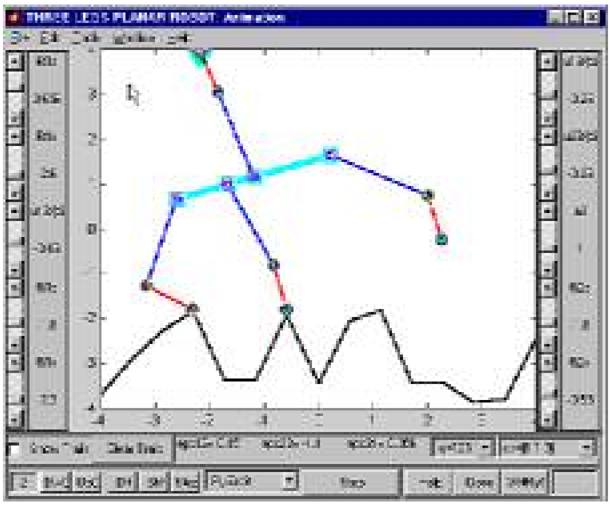


Fig.9. The state of the robot for
$u^{1,j} = ct\,; R_i^x = -2,6\,;\ R_i^z = ct.$

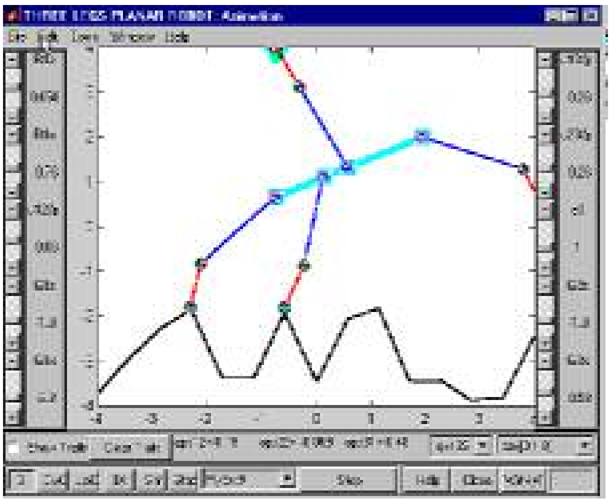


Fig.10. The state of the robot for
$u^{1,j} = ct\,; R_i^x = -0,76\,;\ R_i^z = ct.$

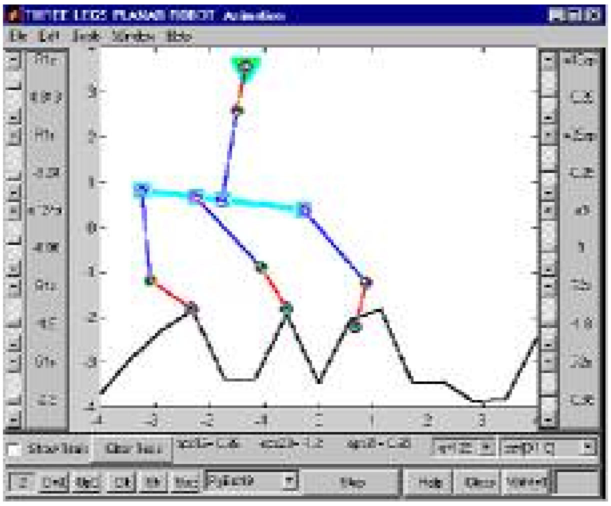


Fig.10. The state of the robot for
$u^{1,j} = ct\,; R_i^x = ct\,;\ R_i^z = 0,818\,.$

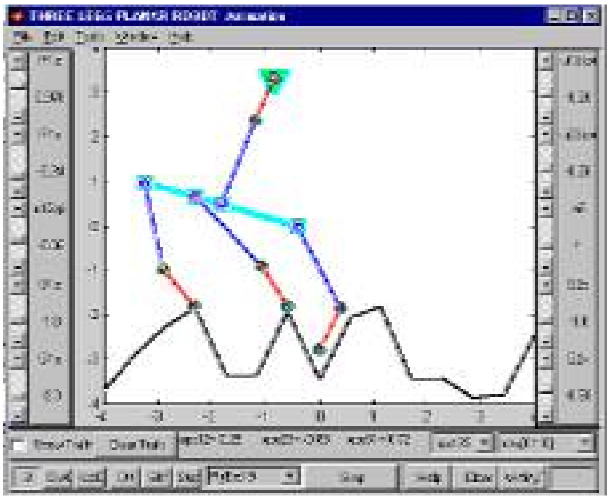


Fig.10. The state of the robot for
$u^{1,j} = ct\,; R_i^x = ct\,;\ R_i^z = 0,978\,.$

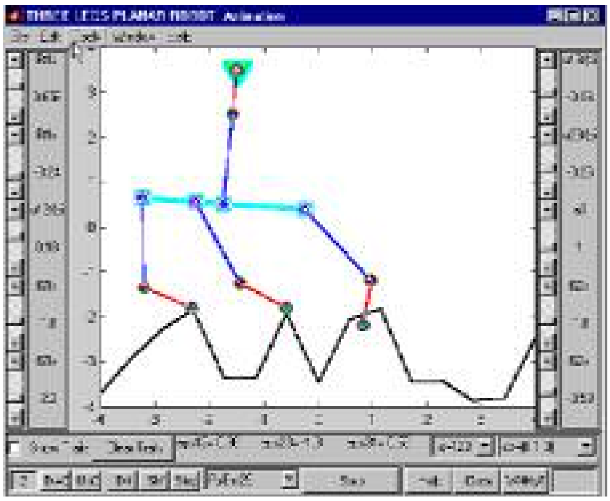


Fig.10. The state of the robot for
$u^{1,j} = 0,18\,; R_i^x = ct\,;\ R_i^z = ct\,.$

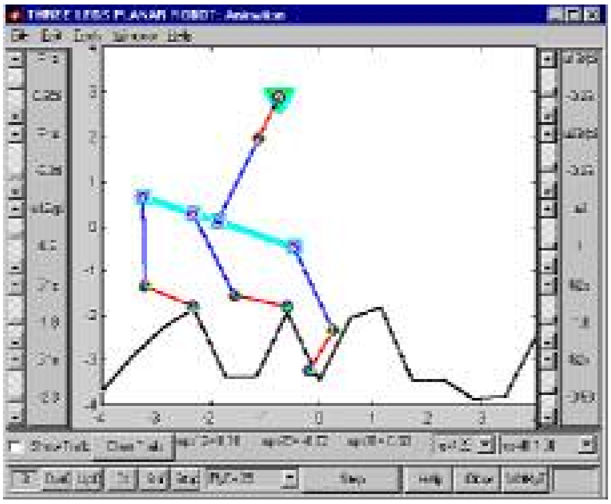


Fig.10. The state of the robot for
$u^{1,j} = 0,3$; $R_i^x = ct$; $R_i^z = ct.$

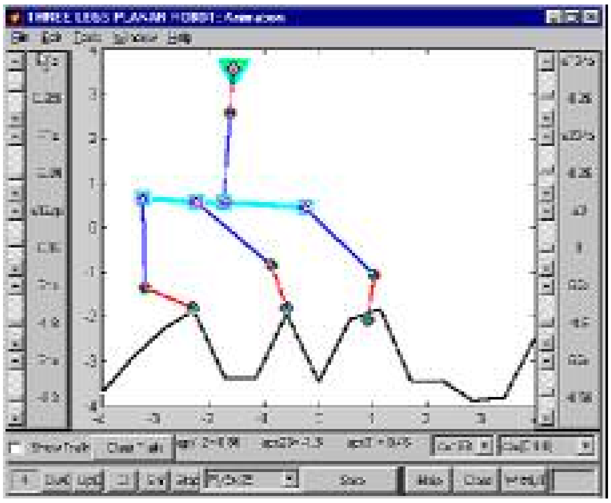


Fig.10. The state of the robot for
$u^{1,j} = -0,16$; $R_i^x = ct$; $R_i^z = ct.$

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