A POINT OF VIEW IN DISCRETE-TIME PARAMETRIC RESONANCE

Vladimir Răsvan

Dept. of Automatic Control, University of Craiova,
A.I.Cuza, 13, Craiova, RO-200585, Romania
e-mail: vrasvan@automation.ucv.ro

Abstract: The parametrically excited systems are systems with periodically varying coefficients. In these systems the phenomenon of parametric resonance is to be met, being characterized by a continuous spectrum composed of several small intervals which tend each to a *critical frequency* when the amplitude of the parametric excitation tends to 0. Also the amplitude of the oscillation around a critical frequency grows exponentially (instead of polynomially – the case of standard resonance).
In the discrete time case the results do not migrate *mutatis–mutandis* from the continuous time one; this paper shows that there are at least two ways of defining critical frequencies in discrete–time parametric resonance.

Keywords: parametric resonance; discrete–time; critical frequencies; Hamiltonian systems.

1. INTRODUCTION

In order to make clear the basic notions we shall discuss only the simplest case, following the line of the basic references (Krein, 1955; Yakubovich and Staržinskii, 1987). Consider S_0 a dynamical system with constant parameters which has oscillatory bounded trajectories. A typical example is

$$\ddot{y} + P_0 y = 0 \tag{1}$$

where y is a m–dimensional vector of the generalized coordinates and $P_0 > 0$ is a real positive definite matrix. Therefore it has positive and simple (or of simple type) eigenvalues; these eigenvalues are ω_j^2 and ω_j are called the eigenfrequencies of system (1).

The motion of the system under ε–parametric excitation – in the linearized formulation – is usually described by

$$\ddot{y} + P(\omega t, \varepsilon) y = 0 \tag{2}$$

where $P : \mathbb{R} \times [0, \varepsilon_0] \to \mathbb{R}^{m \times m}$ is continuous, symmetric and 2π–periodic with respect to the first argument, uniformly with respect to the second one. Also

$$\lim_{\varepsilon \to 0_+} \int_0^{2\pi} |P(t, \varepsilon) - P_0| dt = 0 \tag{3}$$

Definition 1. We call ω a critical frequency for a given ε-parametric excitation of (1) if there is no $\varepsilon_\omega > 0$ such that the motion of (2) is totally stable for $0 < \varepsilon < \varepsilon_\omega$ i.e.such that all solutions of (2) are bounded on $\mathbb{R}$ for $0 < \varepsilon < \varepsilon_\omega$.

Since a change of the time scale $\omega t \rightsquigarrow t$ leads to the modified system

$$\ddot{y} = \frac{1}{\omega^2} P(t, \varepsilon) y = 0 \tag{4}$$

the parametgric resonance is obviously connected with the theory of λ-zones of total stability which has more than 100 years and goes back to Žukovskii(1891/1893) and Liapunov(1899) – for the scalar case. In the vector case we hold the classical result of M.G.Krein (1955) on critical frequencies

Theorem 2. Regardless of the character of the ε-parametric excitation of (1) (i.e. regardless of the choice of $P(t,\varepsilon)$ with the above properties) its critical frequencies can only be the numbers

$$\omega_{j,k,\ell} = (\omega_j + \omega_k)/\ell \ , \ 1 \leq j,k \leq m \ ; \quad \ell = 1,2,\ \ldots \tag{5}$$

We have already mentioned the fact that parametric resonance and the theory of λ-zones are in close connection. In fact there is a long line that unites the pioneering papers of Žukovskii and Liapunov, the paper of Krein (1955) and further research summarized in the two monographs by Yakubovich and Staržinskii (1972,1987). On the other hand we may cite here also a long list of papers on differential equations with periodic coefficients due to various authors, list that may be found in another reference monograph due to Cesari (1963) where this research is integrated.

Generally speaking, the results valid for differential equations can migrate to the field of discrete time systems. Starting from this point of view, Professor Aristide Halanay (1924-1997) has elaborated in Summer 1997 a Research Programme concerning the theory of λ-zones for discrete-time Hamiltonian systems. The accomplishment of this programme is advancing (Halanay and Răsvan, 1999; Răsvan,2000; Răsvan,2002; Răsvan,2003a; Răsvan,2003b). The aim of the present paper is to deal with the associated problem – the parametric resonance for discrete time systems.

2. THE FIRST PROBLEM STATEMENT. A MOST RECENT RESULT ON CRITICAL FREQUENCIES

The first problem statement strongly relies on the following result due to Halanay and Wexler (1968), see also (Halanay and Răsvan, 2000)

Theorem 3. Consider the system

$$y_{k+1} - (A + \varepsilon P_k)y_k + y_{k-1} = 0 \tag{6}$$

with $A = A^*$, $P_k = P_k^* = P_{k+N}$ and assume that the eigenvalues of A are distinct and located inside the disk of radius 2. Then all solutions of (6) are bounded provided $|\varepsilon|$ is small enough.

This is obviously a canonical system having a symplectic matrix; it is obtained as a small perturbation of a canonical system with constant coefficients; also $A + \varepsilon P_k \to A$ for $\varepsilon \to 0$. Clearly $\theta = 2\pi/N$ may be viewed as the frequency of the parametric excitation.

We may now consider the discrete time analogue of (1), namely the system

$$y_{k+1} - 2y_k + y_{k-1} + P_0 y_k = 0 \tag{7}$$

under the following basic assumptions : a) the matrix P_0 is symmetric and positive definite; b) $0 < \omega_j < 2$, $j = 1,\ldots,m$ where ω_j^2 are the eigenvalues of P_0. Since $P_0 > 0$ these eigenvalues are simple or of simple type (with Jordan cells of dimension 1). The characteristic numbers of (7) are in complex conjugate pairs and belong to the unit circle i.e. have moduli equal to 1:

$$\lambda_{1,2}^j = \frac{1}{2}(2 - \omega_j^2 \pm \jmath\omega_j\sqrt{4-\omega_j^2}) = \cos\theta_j \pm \jmath\sin\theta_j = \exp(\pm\jmath\theta_j) \tag{8}$$

It follows that (7) is totally stable. Its ε-parametrically excited associate is

$$y_{k+1} - 2y_k + y_{k-1} + P_k(\varepsilon)y_k = 0 \tag{9}$$

with $P_k(\varepsilon)$ being symmetric, N-periodic and such that

$$\lim_{\varepsilon\to 0}\sum_0^{N-1}|P_0 - P_k(\varepsilon)| = 0 \tag{10}$$

the number $\theta = 2\pi/N$ being the frequency of the parametric excitation. We may state

Theorem 4. If system (7) with eigenfrequencies $\theta_1,\ldots,\theta_m$ defined by (8) is considered subject to a ε–parametric excitation leading to (9), its critical frequencies can only be the numbers $\theta_{j,\ell,q} = (\theta_j + \theta_\ell)/q$, $0 \leq j,\ell \leq m$, $q = 1,2,\ldots$

The theorem has been stated in (Răsvan, 2003) and the proof is to be found there, being an adaptation of the proof due to Krein (1955). It strongly relies on the properties of the monodromy matrices for periodic discrete time Hamiltonian systems (Halanay and Răsvan, 1999) and we shall not insist on technical details.

3. A SECOND STATEMENT OF THE PROBLEM

In order to motivate the second statement, we turn back to the basic references (Krein, 1955; Yakubovich and Staržinskii, 1987) and more precisely to the description given in Section 3 of the paper, formulae (1)–(4). In this case the parametric excitation is given by $P(\omega t,\varepsilon)$ where $\omega > 0$

is the frequency of the parametric excitation that could be some arbitrary real number. At the same time $P(\cdot,\varepsilon)$ is assumed 2π–periodic i.e. it has some standard period.

On the other hand, a main source for discrete time systems is discretization of continuous time ones; but the discrete sequence resulting from discretizing a periodic function is, generally speaking, *almost periodic* unless the discretizing step is an entire or rational submultiple of the period. Or, in this case, it seems reasonable to consider this special discretization as connected to the standard period. Consequently we have to consider first the passage from (2) to (4) and then discretize (4) with the step $2\pi/N$. This will give the discrete parametrically excited system

$$y_{k+1}-2y_k+y_{k-1}+\omega^{-2}P_k(\varepsilon)y_k=0 \quad (11)$$

with $P_k(\varepsilon)$ being symmetric, N–periodic and satisfying

$$\lim_{\varepsilon\to 0_+}\sum_{0}^{N-1}|P_k(\varepsilon)-P_0|=0 \quad (12)$$

which is more alike (3). This point of view requires a re-formulation of *Theorem 4*. In order to state this reformulation we remark first that

$$\sum_{0}^{N-1}|\omega^{-2}P_k(\varepsilon)-\omega^{-2}P_0|=\omega^{-2}\sum_{0}^{N-1}|P_k(\varepsilon)-P_0|$$

and (12) holds for the pair $\omega^{-2}P_0,\omega^{-2}P_k(\varepsilon)$. One has to examine the "intermediate" system

$$y_{k+1}-2y_k+y_{k-1}+\omega^{-2}P_0y_k=0 \quad (13)$$

Its eigenfrequencies are defined starting from the total stability condition for (13) which reads now

$$0<\omega_j/\omega<2\ ,\ j=1,2,\ldots,m$$

This gives a limitation over the number ω which stays for the excitation frequency that is $\omega>(1/2)\max_j\{\omega_j\}$. If ω does not respect this inequality, the comparison principle used in determining the critical frequencies is no longer applicable. Indeed, the mathematical construction in the line of Krein (1955) goes as follows. Assuming that ω is such that (13) is totally stable, we may define the eigenfrequencies of (13) as the numbers $\theta_\ell^\omega\in(0,\pi)$ determined from the equalities

$$2\sin\frac{\theta_\ell^\omega}{2}=\frac{\omega_\ell}{\omega}\ ,\ \ell=1,\ldots,m$$

(Remark that the restriction on ω makes possible to define θ_ℓ^ω as above). The characteristic numbers of (13) are $\exp(\pm\jmath\theta_\ell^\omega)$ and are located on the unit circle. The transition matrix of (13), denoted as $U_k^{\omega,0}$, $k=0,1,2,\ldots$ is of stable type i.e.it has the eigenvalues $\exp(\pm\jmath k\theta_\ell^\omega)$, $\ell=1,\ldots,m$, located on the unit circle and simple or of simple type. Also its eigenvectors are of definite type in the sense of Krein. More precisely

$$\jmath(J\xi_\omega^{(\ell)},\xi_\omega^{(\ell)})=-2\sin\theta_\ell^\omega\neq 0\ ,\quad \ell=\pm1,\ldots,\pm m \quad (14)$$

with $\xi_\omega^{(\ell)}$ – the eigenvector associated to θ_ℓ^ω and

$$J=\begin{pmatrix}0 & I_m\\ -I_m & 0\end{pmatrix}$$

The situation that has to be avoided is that an eigenvalue of the type $\exp(\jmath\theta_\ell^\omega)$ would equal an eigenvalue of the type $\exp(-\jmath\theta_\ell^\omega)$; this would imply that the eigenspace of this eigenvalue contains two vectors of different types in the sense of (13). Therefore

$$(\theta_i+\theta_\ell)k\neq 0\ (mod\ 2\pi)$$

We turn now to system (11) which is canonical like 13) but it is N–periodic. Its stability properties are determined by its multipliers – the eigenvalues of the monodromy matrix $U_N^{\omega,\varepsilon}$ of system (11). Let us view (13) as N–periodic : its monodromy matrix would be $U_N^{\omega,0}=U_k^{\omega,0}|_{k=N}$. From (12) we deduce, following the results of Krein (1955) or (Halanay and Wexler, 1968) that

$$\lim_{\varepsilon\to 0_+}U_N^{\omega,\varepsilon}=U_N^{\omega,0}$$

uniformly with respect to ω and for each $N\geq 1$. Moreover, if $U_N^{\omega,0}$ is of stable type (with eigenvalues located on the unit circle, with definite eigensubspaces), the neighboring matrices are such hence $U_N^{\omega,\varepsilon}$ are of stable type for ε small enough $(0<\varepsilon<\varepsilon_\omega)$.

Critical frequency means instability for any small $\varepsilon>0$ i.e. non-existence of ε_ω as above. But this may happen only if $U_N^{\omega,0}$ *is not of stable type.* It follows that there should exist some i and ℓ such that

$$(\theta_i^\omega+\theta_\ell^\omega)N=2q\pi$$

that is

$$\theta_i^\omega+\theta_\ell^\omega=\frac{2q\pi}{N}$$

We may use now the definition of θ_i^ω in order to obtain, after simple trigonometric manipulation, the following result

Theorem 5. Consider system (7) with the eigenfrequencies $\theta_1,\ldots,\theta_m$ defined by (8) i.e. starting from the eigenvalues $\omega_1^2,\ldots,\omega_m^2$ of P_0 and assume this system to be subject to a ε–parametric excitation leading to (11) with $\omega > (1/2)\max_\ell\{\omega_\ell\}$. then its critical frequencies can only be the numbers

$$\omega_{i,\ell,q} = \frac{1}{2}\left|\frac{\omega_i + (W_N)^{q/2}\omega_\ell}{\Im(W_N)^{q/2}}\right| \ , 1 \le i, \ell \le m \ ; \quad q = 0, 1, \ldots, 2(N-1) \tag{15}$$

where W_N has the significance given in Digital Signal Processing i.e. $W_N = \exp(-2\pi \jmath/N)$.

An interesting remark is that within this view of parametric resonance *the number of the critical frequencies is finite.*

4. CONCLUSIONS

In this paper there are presented two models of the parametric resonance in discrete time systems. In both cases there was considered the parametric excitation of the canonical system described by the Euler–like discretization of the matrix Hill equation and in both cases there were given explicit formulae for the critical frequencies. A special mention for the condition on the excitation frequency in the second model i.e. $\omega > (1/2)\max_\ell\{\omega_\ell\}$. Denoting $\lambda = \omega^{-1}$ the inequality becomes

$$\lambda^2 < 4(\max_\ell\{\omega_\ell\})^{-2} = 4|P_0|^{-1} = \\ = \frac{4}{N}\left(\sum_0^{N-1}|P_0|\right)^{-1} \tag{16}$$

which is exactly the *Liapunov–like estimate* for the central stability zone in the corresponding case (Răsvan, 2002) provided the matrix norm is taken the largest singular eigenvalue of P_0.

Finally let us remark that the validation of the right model among the two (or of the both ones) will take place by further research on parametric resonance along the lines of the (by now) classical reference (Yakubovich and Staržinskii, 1987).

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