# SCHWARTZ DISTRIBUTION PROPERTIES
# IN DIGITAL CONTROL SYSTEMS APPLICATIONS

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**Abstract:** In this paper Schwartz distributions, their properties and digital control applications are considered. Using the definition of distribution, the mathematical formulas for discrete elements are presented in order to determine the mathematical equations of digital control systems. This approach allows describing the hybrid systems containing analog and digital elements with the common equations on distributions. The equations of distributions are very simple comparing to the equations on normal functions. Based on distribution equations there can be designed more simple, flexible and reliable systems. The paper describe the specific properties that can be used in digital control and design the models. Based on this theory the complex PWM-control of power electronics inverter application is presented. The results are similar to classical PWM inverter produced by electronic components, but more simple.

**Keywords**: distributions, digital, control, applications

## 1. INTRODUCTION

A lot of phenomena cannot be represented as a function or as a mathematical formula because of its discontinuities, severe non-linearity, non-derivation etc.

In the real function theory this regions are avoided with the partitioning method, eliminating the discontinuities and never using the derivation operator in critical points.

This approach has two disadvantages: first, the representation is complicated and unusual and second, just the discontinuities are eliminated where the system works (digital systems, commutations relay, logical transducers and hybrid elements).

It is possible to extend the function theory using the Schwartz distributions, which generalizes the mathematical operators and can be applied on the whole domain. On the other hand by this method is possible to define new distributions not existing in the real function class, but very usual in the phenomena analyzing.

Now, consider the most usual "function" sign(x) which returns the sign of the argument (fig.1).

$$\mathrm{sign}(x) = \begin{cases} 1; & x \geq 0 \\ -1; & x < 0 \end{cases}$$

$$[\mathrm{sign}(x)]' = \begin{cases} 0; & x \neq 0 \\ 2 \cdot \delta(x); & x = 0 \end{cases}$$

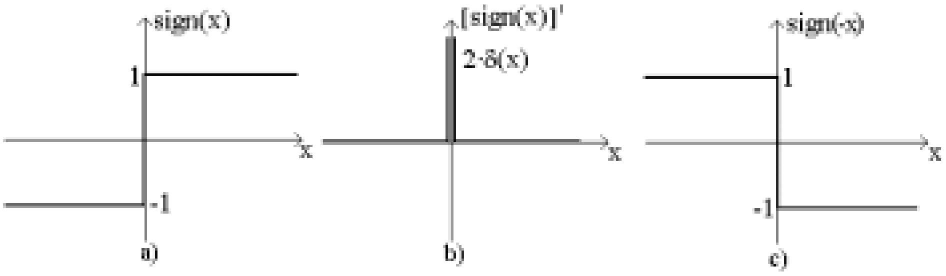



Fig.1. a) Sign distribution; b) Sign derivation; c) Sign invert

If sign(x) is treated as a function, its derivate $[\mathrm{sign}(x)]' = 0$, but this is not true because of the origin discontinuity which produce a short pulse. It is possible to define: $\mathrm{sign}(-x) = -\mathrm{sign}(x)$ as an odd function.

Switching a switch is a very known phenomenon, represented as a "function" $\theta(x)$ (fig.2).

$$\theta(x) = \begin{cases} 1; & x \geq 0 \\ 0; & x < 0 \end{cases} \qquad \theta(-x) = \begin{cases} 1; & x \leq 0 \\ 0; & x > 0 \end{cases}$$

$$[\theta(x)]' = \delta(x) \qquad [\theta(-x)]' = -\delta(x)$$

$$\int_{R^+} \theta(x)dx = \theta(x) \cdot x = x_+ = r(x)$$

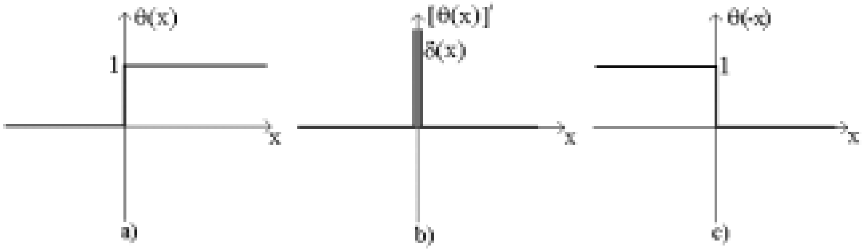


Fig.2. a) Heaviside distribution; b) Heaviside derivation; c) Heaviside invert

As above, if $\theta(x)$ is a function then $[\theta(x)]' = 0$ and if it is a distribution then $[\theta(\pm x)]' = \pm\delta(x)$. The relations between these distributions are:

$$\theta(x) + \theta(-x) = 1; \quad sign(x) = \theta(x) - \theta(-x) = 2\cdot\theta(x) - 1$$

Based on this simple distribution is possible to define new distributions. For example the memory as a window behavior or the rectifier of AC voltage can be represented very simple as a distribution.

In case of the hystheresis distribution $h(x)$ (fig.3.a) the classical definition is complicated as presented below:

$$h(x) = \begin{cases} \left.\begin{matrix} 1; & x \geq a \\ -1; & x < a \end{matrix}\right\} \text{if } \frac{dx}{dt} \geq 0 \\ \left.\begin{matrix} 1; & x \geq -a \\ -1; & x < -a \end{matrix}\right\} \text{if } \frac{dx}{dt} \leq 0 \end{cases}$$

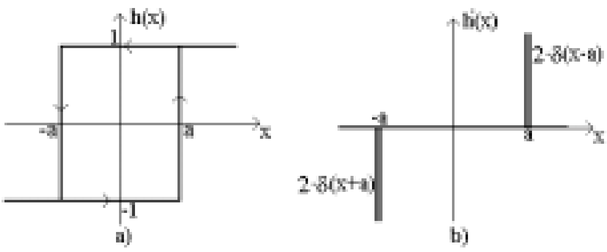


Fig.3. a) Hystheresis distribution; b) Hystheresis derivation

As distribution $h(x)$ can be written:

$$h(x) = sign(x-a)\cdot\theta\left(\dot{x}\right) + sign(x+a)\cdot\theta\left(-\dot{x}\right)$$

$$h(x) = [2\cdot\theta(x-a) - 1]\cdot\theta\left(\dot{x}\right) + [2\cdot\theta(x+a) - 1]\cdot\theta\left(-\dot{x}\right)$$

Based on relations:

$$\theta(x-a)\cdot\theta\left(\dot{x}\right) + \theta(x+a)\cdot\theta\left(-\dot{x}\right) = \theta\left(x - a\cdot sign\left(\dot{x}\right)\right)$$

$\theta\left(\dot{x}\right) + \theta\left(-\dot{x}\right) = 1$, results a new formula:

$$h(x) = sign\left(x - a\cdot sign\left(\dot{x}\right)\right)$$

If is calculated $[h(x)]'$ results (fig.3.b):

$$[h(x)]' = \left[x - a\cdot sign\left(\dot{x}\right)\right]'\cdot\theta'\left[x - a\cdot sign\left(\dot{x}\right)\right] =$$

$$= \dot{x}\cdot\delta\left[x - a\cdot sign\left(\dot{x}\right)\right] = \pm\delta[x \mp a]$$

In the case of the rectifier function $d(x)$ applied for $\sin(x)$ (fig.4) can be observed that the positive wave is $\sin_+(x) = \theta[\sin(x)]\cdot\sin(x)$ and the negative wave is $\sin_-(x) = \theta[-\sin(x)]\cdot\sin(x)$.

If we combine these two distributions results:

$$\sin(x) = \sin_+(x) + \sin_-(x)$$

$$d(x) = \sin_+(x) - \sin_-(x) = sign[\sin(x)]\cdot\sin(x)$$

## 2. DISTRIBUTIONS AND THEIR PROPERTIES FOR DIGITAL CONTROL APPLICATIONS

We consider the n-dimensional Euclidian real space $R^n$, organized as a vectorial space. The elements of this space are n-coordinates vectors: $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ and scalar product and vector norm is defined as follows:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \cdot y_i; \qquad |x| = \sqrt{\sum_{i=1}^{n} x_i^{\,2}};$$

$$d(x, y) = |x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

On this space will be considered the functions: $F(x): R^n \to R$. From this functions will be selected $\varphi(x)$ which are of n order, continuous, derivable and also the derivates are continuous and have zero value out of their domain. The domain of these functions is closed and compact.

The space of $\varphi(x)$ functions continuous and derivable of n order is $K^n$ and has the properties:

- $\forall\varphi_1(x), \varphi_2(x) \in K^m,\ \alpha, \beta \in R^n$ ; $: \alpha\cdot\varphi_1(x) + \beta\cdot\varphi_2(x) \in K^n$
- $\forall\varphi_k(x) \in K^m : \lim_{k\to\infty}|\varphi_k(x)| = 0$.

It is possible to define the functional applications between $K^m$ and $R^n$ as follows:

$$f : K^m \to R^n : \ f[\varphi(x)] = c.$$

Definition: Distributions are real applications as a result of linear and continuous functions on the $K^m$ space.

The most important distributions in digital systems are: pulse, step and ramp distributions.

The pulse distribution or Dirac pulse is defined as:

$$\delta : K^m \to R^n : \ \delta[\varphi(x)] = \varphi(0) = \begin{cases} 0; & x \neq 0 \\ -\infty; & x = 0 \end{cases};$$

$$\varphi(x) \in K^m$$

$$\int_R \delta[\varphi(x)] dx = 1$$

This distribution represents an origin concentrated infinite pulse, which has a weigh equal to 1 (fig.4.a).

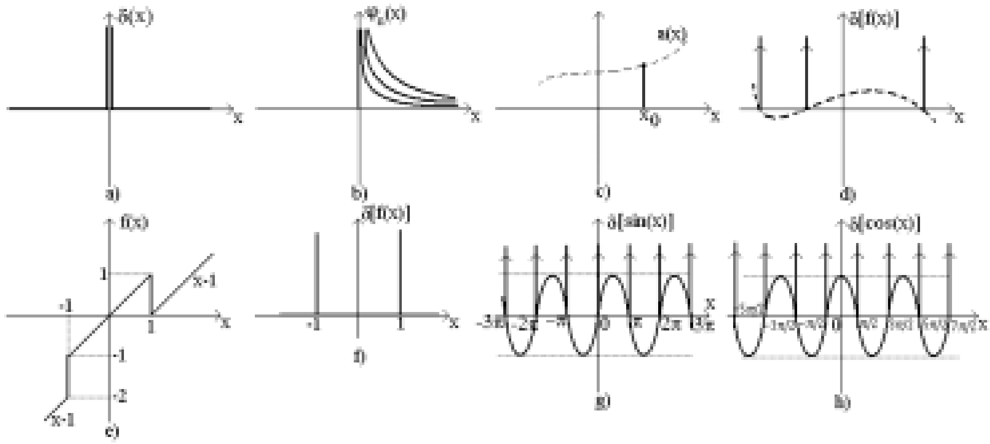


Fig.4. Dirac distribution and properties

The most important properties of Dirac distribution are:

1. $\delta(x - x_0) = \varphi(x_0)$
2. $a(x) \cdot \delta(x - x_0) = a(x_0) \cdot \delta(x - x_0)$
3. $\delta(\alpha \cdot x + \beta) = \frac{1}{|\alpha|} \cdot \delta\left(x + \frac{\beta}{\alpha}\right)$
4. $\delta(-x) = \delta(x)$
5. $k \cdot \delta(x - x_0) = k \cdot \varphi(x_0) = \begin{cases} 0; & x \neq x_0 \\ k; & x = x_0 \end{cases}$
6. $\delta[f(x)] = 0; \quad f(x) \neq 0$
7. $\frac{d[f(x)]}{dx} = f(x)' + \sum_{i=1}^{p} s_i \cdot \delta(x - x_i);$
   $s_i = f(x_i + 0) - f(x_i - 0)$; $x_i$ first order discontinuities. For example:
   $$f(x) = \begin{cases} x; & -1 < x < 1 \\ x - 1; & x < -1 \text{ or } x > 1 \end{cases};$$
   $$\frac{d[f(x)]}{dx} = 1 + \delta(x + 1) - \delta(x - 1).$$
8. $\delta[f(x)] = \sum_{i=1}^{p} \frac{1}{|f'(x)|} \cdot \delta(x - x_i)$; where $x_i$ are simple solutions of $f(x) = 0$ equation.
9. $\delta[\sin(x)] = \sum_{k=-\infty}^{\infty} \delta(x - k \cdot \pi); \ k \in Z$
10. $\delta[\cos(x)] = \sum_{k=-\infty}^{\infty} \delta\left(x - (2 \cdot k + 1) \cdot \frac{\pi}{2}\right); \ k \in Z$

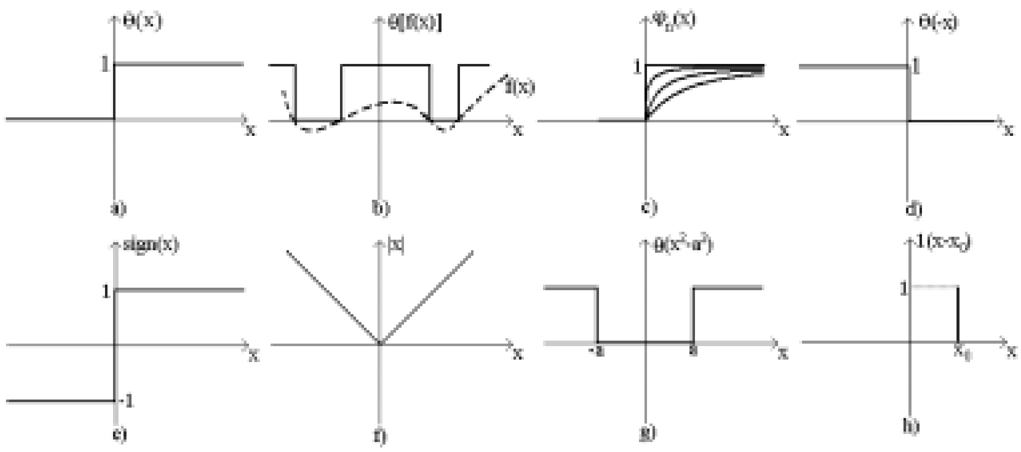


Fig.5. Heaviside distribution and properties

The step distribution or Heaviside distribution is defined as follows:

$$\theta : K^m \to R; \qquad \theta[\varphi(x)] = \begin{cases} 1; & x \geq 0 \\ 0; & x < 0 \end{cases}$$

This distribution (fig.5.a) is a limit of $\varphi_n(x) = 1 - e^{-n\cdot x}$ (fig.5.b). The important properties of $\theta(x)$ are:

1. $\theta(-x) = 1 - \theta(x)$
2. $\theta(x) - \theta(-x) = \text{sign}(x)$
3. $\theta(x) + \theta(-x) = 1$
4. $x \cdot \text{sign}(x) = |x|$
5. $\theta(x^2 - a^2) = 1 - \theta(x+a) + \theta(x-a)$
   $\theta(a^2 - x^2) = \theta(x+a) - \theta(x-a)$
6. $\theta(x) \cdot f(x) = f_+(x)$
7. $\theta[\varphi(x - x_0)] = 1(x - x_0)$

The ramp distribution is defined as follows:

$$r : K^m \to R; \qquad r[\varphi(x)] = \begin{cases} x; & x \geq 0 \\ 0; & x < 0 \end{cases}; \qquad r(x) = x_+$$

The ramp is a combination between function $f(x) = x$ and step distribution: $r(x) = x \cdot \theta(x) = x_+$

The ramp distribution and its properties are in fig.6.

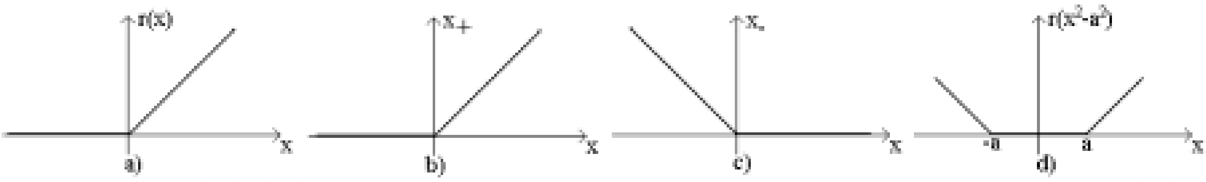



Fig.6. Ramp distribution and properties

The most important properties of ramp distribution are:

1. $-x \cdot \theta(-x) = x_-$ (negative ramp)
2. $x_+ - x_- = x$
3. $|x| = x \cdot \text{sign}(x) = x_+ + x_-$ (modulus)
4. $\frac{1}{2} \cdot (x + |x|) = x_+$
5. $r(x^2 - a^2) = \theta(x-a) \cdot (x-a) + \theta(x+a) \cdot (x+a)$
6. $[x_+]' = \theta(x)$
7. $[x_-]' = -\theta(-x)$

## 3. SIMULATION AND SOFTWARE IMPLEMENTATION

First the model of $\delta(t), \theta(t), r(t), \text{sign}(t)$ and $|t|$ were designed and simulated.

Based on these models in fig.7 was designed the model for the following distributions: $\delta[\sin(\omega \cdot t)], \delta[\cos(\omega \cdot t)], \delta[x^2 - 4], \theta[x^2 - 4], \text{sign}[\sin(\omega \cdot t)]$ and then simulated.

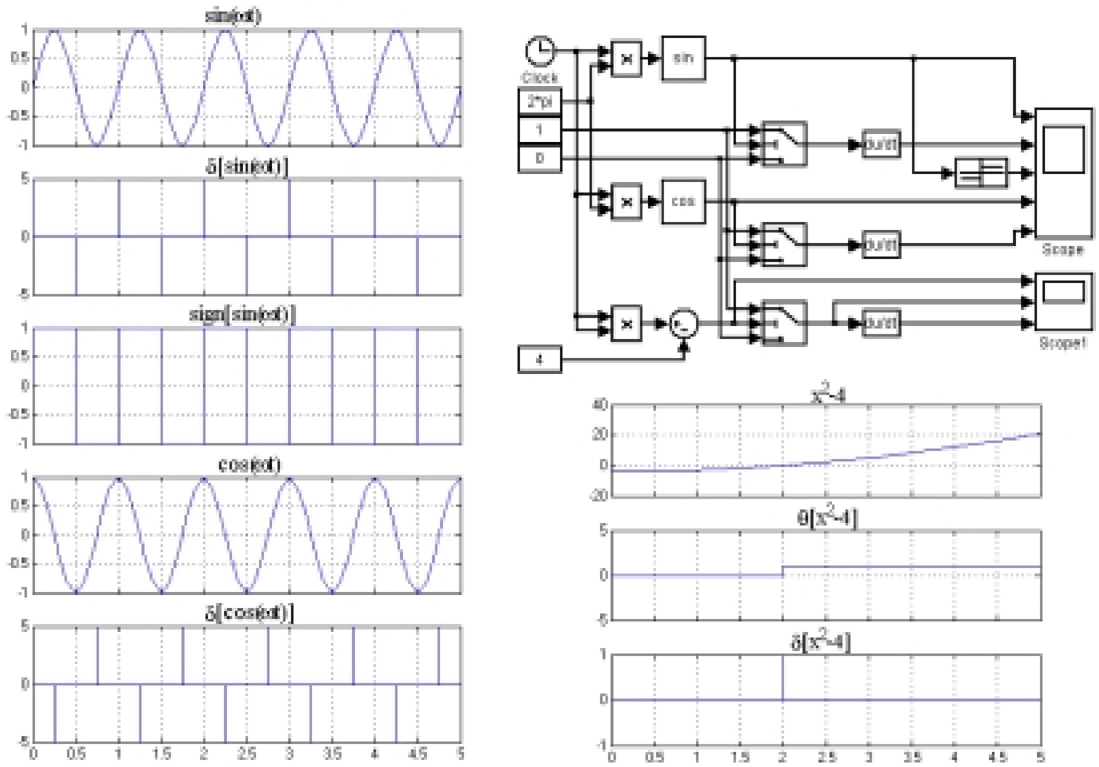



Fig.7. Simulation of $\delta[\sin(\omega \cdot t)], \delta[\cos(\omega \cdot t)], \delta[x^2 - 4], \theta[x^2 - 4], \text{sign}[\sin(\omega \cdot t)]$

The software was written in assembly language for elementary distributions: $\delta(t), \theta(t)$ and $r(t)$.

In fig.8 is presented the subroutine for $\theta(t)$ and the results on the screen after running.

```
grafic_treapta proc near
; based on the ramp
distribution determined
previously is calculated the
step distribution
    mov di,offset RampComplet
     inc di
     inc di
    mov si,offset StepComplet
     mov word ptr [si],0
     inc si
     inc si
; step is graphic displayed
     mov x1,350
     mov y1,20
     mov x2,435
     mov y2,20
     mov culin,29
     call xilinie
     mov x1,430
     mov x2,430
     mov y2,700
     mov culin,29
     call xilinie
; step is saved in the
StepComplet buffer
     mov x1,430
     mov y1,20
     mov culin,32
     mov index,1
repTr: mov ax,[di]
     cmp ax,0
     je Tzero
     mov word ptr [si],1
     inc si
     inc si
     mov ax,index
     mov y2,20
     add y2,ax
     mov x2,360
     jmp trT
Tzero: mov word ptr [si],0
     inc si
     inc si
     mov ax,index
     mov y2,20
     add y2,ax
     mov x2,430
trT: call xilinie
     mov ax,x2
     mov x1,ax
     mov ax,y2
     mov y1,ax
     inc word ptr index
     inc di
     inc di
     cmp index,700
     jae outT
     jmp repTr
outT: ret
grafic_treapta endp
```

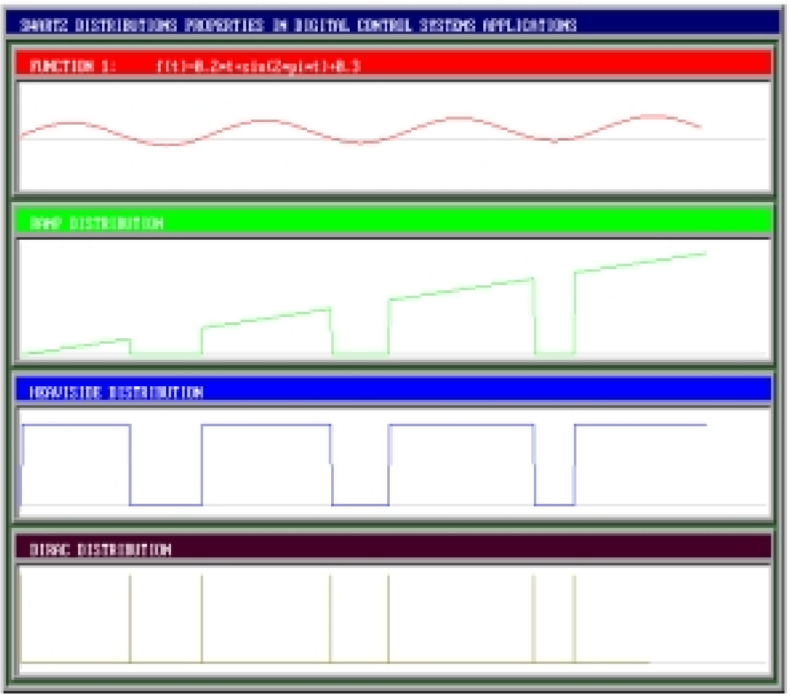



Fig.8. Software for elementary distributions

## 4. EXPERIMENTAL RESULTS FOR POWER INVERTER CONTROLLER

The power inverter controller must produce six appropriate PWM pulses that are 120 degrees unphased. This is a complicated process because there must be solved by approximations the following nonlinear transcendental equations to determine the pulses (fig.9).

$$R \cdot \sin\left(\alpha + \frac{2 \cdot (i-1) \cdot \pi}{3}\right) =$$

$$= (-1)^{k-1} \cdot \frac{2 \cdot p}{\pi} \cdot \left(\alpha + \frac{2 \cdot (i-1) \cdot \pi}{3}\right) + \frac{(k-1) \cdot \pi}{p}$$

where: i is the phase number, p is the frequency index, R is the modulation index and k is the pulse number.

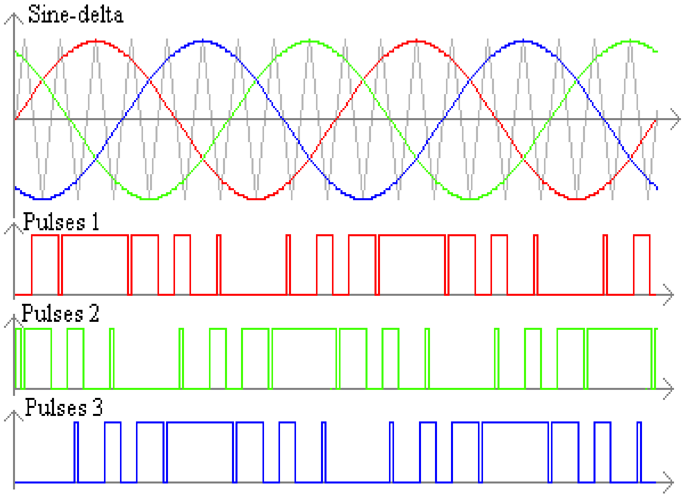



Fig.9. PWM pulses

Using the distributions theory this problem can be solved very simple and precisely.

First the rectangular distribution will be generated by formula: $\theta[\sin(\omega \cdot t)] = \sum_{i=0}^{2 \cdot p} \delta(t - k \cdot \pi)$ ·

After this, using the integration, applying $\delta(t)$ the pulses will be obtained (fig.10).

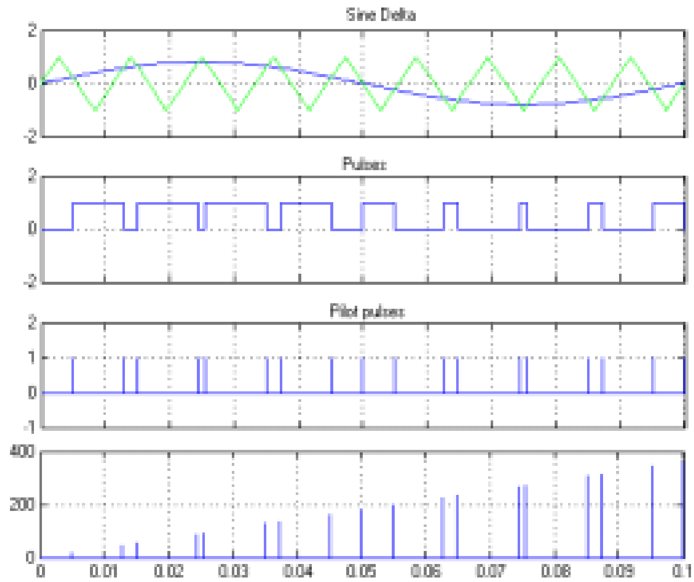
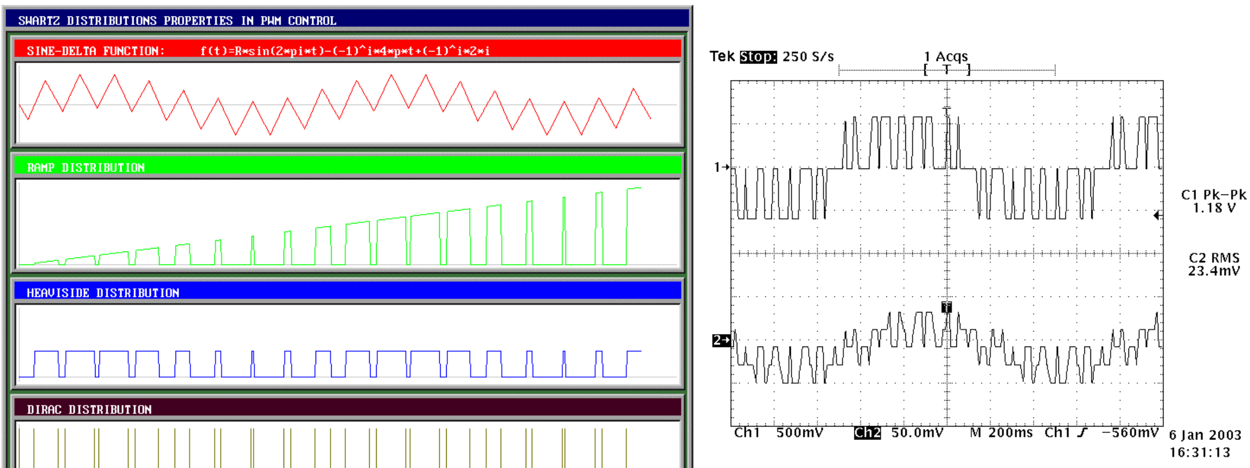



Fig.10. Simulation results

The software was written in assembly language. There are different routines. One is to model the nonlinear transcendental equations at sample time and others to apply ramp, step and Dirac distributions to sine-delta function. The step distribution outputs the pulses for one switch and the other five pulses are determined by unphasing it with 120$^o$ and 240$^o$.

This software was tested and compared with the pulses produced by an electronic PWM inverter based on the IPM module FUJI 120RA100 (fig.11).




a) b)

Fig.11. a) PWM software; b) Experimental results

## 5. CONCLUSIONS

1. The distributions are very useful for discreet signals like those in digital control.
2. Using the properties of distributions many complex problems of digital control can be solved easy and precisely.
3. The distributions are suitable for software implementation and by this offer very good support for software oriented solution of digital control applications.
4. The experimental model is simple and the results are very closed and accurate comparing with the classical ones.

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