FORCED-REGIME OSCILLATIONS IN NONLINEAR SERVOMECHANISMS WITH LINEAR DAMPING

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Abstract: The aim of the paper is to present the existence and appearance of the oscillations, based on resonant jumps, in systems described by Duffing-type equation with linear damping. The conditions necessary for jump producing and for oscillations appearance are emphasised.

Key words: Nonlinear system, oscillations, Duffing equation, linear damping

1. INTRODUCTION

The studies presented by (Duffing 1918) have brought the first solving of the non-linear systems and highlighted their importance.

The most important results in this field have been collected in some reference books by several authors (Stoker 1950, Kauderer 1958, Minorsky 1962, Hayashi 1964, Dinca et al. 1969 and Savin et al. 1973).

There is a large category of non-linear systems described by the Duffing-type equation with linear damping. For such systems, in forced regime, where emphasised by various authors response-amplitude discontinuities for frequency or amplitude variation of the excitation-signal. In such non-linear systems, for constant values of the excitation-signal frequency and amplitude, the authors highlight the appearance of the oscillations through resonant jumps.

The oscillations yields when in the system occur resonant jumps and at least one parameter of the system’s equation *p(a)* varies slowly together with the modification by jump of the response amplitude.

2. THEORETICAL CONSIDERATIONS

One considers a positioning system based on a DC motor with separate excitation and fed by an amplifier. There is analysed the case when, before the saturation appearance in the motor inductor’s magnetic circuit, a marked amplifier saturation occurs. In this particular case, the motor in its whole assembly must be considered linear.

Let consider the diagram shown in the Figure 1, where *g(x)* is the nonlinear part and *L* the linear part of the system.

Supposing that the static torque is missing, on the out-going shaft of the system one obtains the following torque equilibrium equation:

$$J\frac{d^2e(t)}{dt^2}+K_f\frac{de(t)}{dt}=K_1y(t) \qquad (1)$$

According the block diagram in Fig.1, the output is

$$e(t)=r(t)-x(t) \qquad (2)$$

and then (1) can be put in the form:

$$J\left(\frac{d^2r(t)}{dt}-\frac{d^2x(t)}{dt^2}\right)+K_f\left(\frac{dr(t)}{dt}-\frac{dx(t)}{dt}\right)=K_1y(t) \qquad (3)$$

or even

$$\frac{d^2x(t)}{dt^2}+\frac{K_f}{J}\frac{dx}{dt}+\frac{K_1}{J}y(t)=\frac{d^2r(t)}{dt^2}+\frac{K_f}{J}\frac{dr(t)}{dt} \qquad (4)$$

The saturation-type of the nonlinearity given in Fig. 2 can approximate through

$$g(x)=x\left(1+\beta\cdot x^2\right) \qquad (5)$$

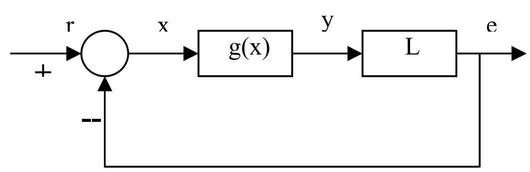

Fig. 1 The control-system structure

where $\beta<0$ and so, the equation (4) becomes:

$$\frac{d^2x(t)}{dt^2}+\frac{K_f}{J}\frac{dx(t)}{dt}+\frac{K_1}{J}x(t)\left(1+\beta x^2\right)= \\ =\frac{d^2r(t)}{dt^2}+\frac{K_f}{J}\frac{dr(t)}{dt} \qquad (6)$$

If

$r(t)=R*\sin\omega\cdot t$

then

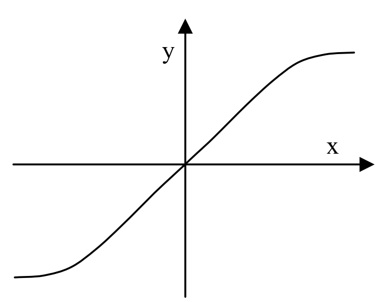

Fig. 2 Saturation of the nonlinearity

$$\frac{K_1}{J}=\omega_0^2 \quad \text{and} \quad \frac{K_f}{J}=2\alpha \tag{7}$$

ω_0 being the self pulsation and α the linear damping. So the equation (6) takes the form known as the Duffing equation:

$$\frac{d^2x(t)}{dt^2}+2\alpha\frac{dx(t)}{dt}+\omega_0^2x(t)\left(1+\beta\cdot x(t)^2\right)= \\ =R\cos(\omega t+\gamma) \tag{8}$$

where the notations are

$$R=R^*\omega\sqrt{\omega^2+4\alpha^2} \quad \text{and} \quad tg\gamma=\frac{\omega}{2\alpha}. \tag{9}$$

The equation (8) describes the forced regime behaviour of the system given in Figure 1. For such a regime many authors (Duffing 1918, Hayashi 1964) have emphasized the amplitude jumps appearance at variation of frequency or excitation signal amplitude.

In such systems can be highlighted the oscillations through resonant jumps if one of the equation coefficients has an inertial variation depending on the amplitude, when R and ω are constants.

In order to respect the solution's stability and oneness conditions it becomes necessary

$$g'(x)=\omega_0^2\left(1+3\beta\cdot x^2\right)>0, \quad \forall x>0 \tag{10}$$

which is fulfilled for

$$|x|<\frac{1}{\sqrt{-3\beta}}$$

Because the exact analytic solutions of the Duffing equation are unknown, the harmonic linearization method is applied. One considers the fundamental term:

$$x(t)=a\cdot\cos\omega t \tag{11}$$

Introducing this in equation (8) one obtains:

$$\left[(\omega_0^2-\omega^2)a+\frac{3}{4}\beta\omega_0^2a^3+R\cos\gamma\right]\cos\omega t- \\ -2\left(2\omega\alpha a+R\sin\gamma\right)\sin\omega t+\frac{1}{4}\beta\omega_0^2a^2=0 \tag{12}$$

that involves

$$(\omega_0^2-\omega^2)a+\frac{3}{4}\beta\omega_0^2a^3+R\cos\gamma=0 \\ 2\omega\alpha a+R\sin\gamma=0 \tag{13}$$

Removing γ between the equations (13) one obtains:

$$\left(\omega_0^2-\omega^2+\frac{3}{4}\beta\omega_0^2a^2\right)^2+4\omega^2\alpha^2=\frac{R^2}{a^2} \tag{14}$$

Transforming relation (14), it can be written in the form:

$$F\left(\frac{R}{\omega_0^2},a\right)=\left(u^2-1+\frac{3}{4}\beta a^2\right)^2+4\omega^2\lambda^2-\frac{R^2}{\omega_0^4}=0 \tag{15}$$

with $\lambda=\frac{\alpha}{\omega_0}$ and $u=\frac{\omega}{\omega_0}$.

The function graph in the $\left(R/\omega_0^2,a\right)$ plane is called resonance curve. For two values β_1 and β_2 this has the aspect presented in Fig. 3.

The geometrical locus of the points where the resonance curve allows horizontal and respectively vertical tangents can be obtained from

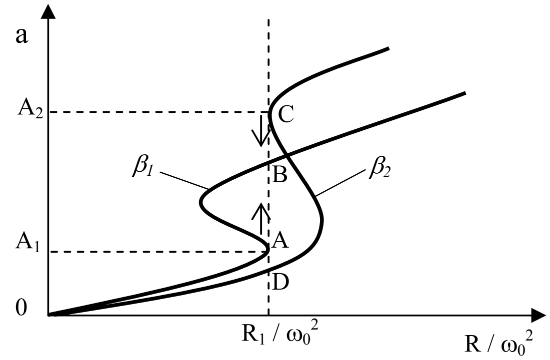

Fig. 3 Explanatory for the resonance jump phenomenon

$$\frac{da}{d\left(\frac{R}{\omega_0^2}\right)}=\frac{\frac{\partial F}{\partial\left(\frac{R}{\omega_0^2}\right)}}{\frac{\partial F}{\partial a}}=\frac{2\frac{R}{\omega_0^2}}{\frac{1}{a}\left[\frac{3}{2}\beta a^2\left(u^2-1+\frac{3}{4}\beta a^2\right)-\frac{R^2}{\omega_0^2a^2}\right]} \tag{16}$$

The geometrical locus of the points where the resonant curves allow horizontal tangents is:

$$\frac{\partial F}{\partial\left[\frac{R}{\omega_0^2}\right]}=0 \tag{17}$$

which involves $R=0$.

The points where the vertical tangents are allowed yields from:

$$\frac{\partial F}{\partial a}=0 \tag{18}$$

and this means that

$$\frac{9}{8}\beta^2a^4+\frac{3}{2}\beta a^2\left(u^2-1\right)-\frac{R^2}{\omega_0^4a^4}=0 \tag{19}$$

From (19) results:

$$\left(\frac{R}{\omega_0^2}\right)^4\frac{4}{9\beta^2a^8}-\left(\frac{R}{\omega_0^2}\right)^2\frac{1}{a^2}+4\omega^2\lambda^2=0$$

Solving (20) in respect of the variable R/ω_0^2 it results:

$$\left(\frac{R}{\omega_0^2}\right)^2=\frac{\frac{1}{a^2}\left(1\pm\sqrt{1-\frac{4\omega^2\lambda^2}{3\beta^2a^2}}\right)}{\frac{8}{9}\frac{1}{\beta^2a^8}} \tag{20}$$

or

$$\left(\frac{R}{\omega_0^2}\right)^2=\frac{9}{8}\beta^2a^4\left(1\pm\sqrt{1-\frac{4\omega^2\lambda^2}{3\beta^2a^2}}\right) \tag{21}$$

This curve exists only for

$$\frac{4\omega^2\lambda^2}{3\beta^2a^2}<1 \quad \text{or} \tag{22}$$

$$a^2>\frac{4\omega^2\lambda^2}{3\beta^2} \tag{23}$$

The increasing or decreasing jump of the amplitude happens when the condition (18) is fulfilled simultaneously with

$$\frac{\partial^2 F}{\partial a^2} > 0 \qquad (24)$$

$$\frac{\partial^2 F}{\partial a^2} < 0 \qquad (25)$$

Let consider that the parameter has an inertial variation with delay, described by a general form as following:

$$\beta(t) = \beta_0 + Ka(t-\tau_3)\left(1 + \frac{T_1}{T_2 - T_1} e^{-\frac{t-\tau_1}{T_1}} + \frac{T_2}{T_1 - T_2} e^{-\frac{t-\tau_2}{T_2}}\right)$$

where $T_1, T_2 >> \frac{2\pi}{\omega}$ and $\frac{T_1 + T_2}{2\sqrt{T_1 T_2}} \geq 1$ (26)

If the condition of the amplitude-jumps appearance is achieved for $R=R_1$, when the parameter $\beta(a)$ has the β_1 value, than occurs an increasing by jump of the response amplitude, from the value that corresponds to the point A to the value that corresponds to the point B (v. Fig.3). At the same time, the parameter varies slowly towards the value β_2, involving a corresponding change of the resonance curve. So, the operating point is moving slowly from B to C. In the moment when the resonance curve become tangent to the point C, the jump condition is again accomplished and the amplitude will decrease up to the value that corresponds to the point D. The parameter β suffers a low variation that determines the slowly displacement of the operating point from D to A, when the process will reiterate.

Thus, in the system happens an oscillation process through resonant jumps, both of the amplitude and phase. The envelope of the excitation signal is a low oscillation.

The variation that corresponds to the phase can be determined with the relations:

$$\sin\gamma = -\frac{2\omega\alpha a}{R} \qquad (27)$$

or

$$\cos\gamma = \frac{a}{R}\left(\omega_0^2 - \omega^2 + \frac{3}{4}\beta\omega_0^2 a^2\right) \qquad (28)$$

3. SIMULATION RESULTS

A confirmation of the discussed phenomenon can be obtained by numerical simulation. Fig.4 presents the simulated resonance curves for several values of the parameter β.

In order to get the system behaviour, a simulating program in the Matlab-Simulink environment was performed. This used a fourth order Runge-Kutta algorithm to solve the Duffing equation.

The oscillations get by resonant jumps for the following values of the parameters:

λ=0.15, β_0=0.02, β_{final}=0.07, ω_0=3rad/sec, ω=5.19 rad/sec, R_1=72, K=0.006

and for different time-constants values are presented in Figs. 5-8.

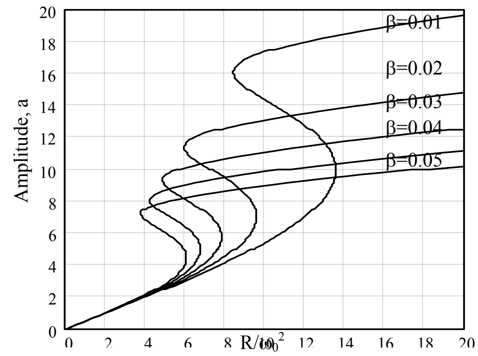

Fig. 4 Resonant curves for different values of the parameter β

4. CONCLUSIONS

The paper emphasised the oscillation's appearance, based on resonant jumps, in systems described by Duffing-type equation with linear damping and nonlinearity having inertial variation with delay.

The conditions necessary for jump producing and for

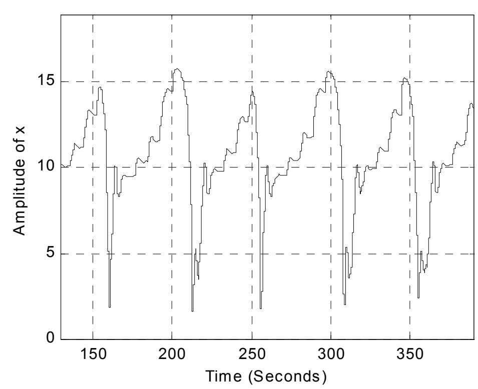

Fig.5 Variable's x oscillations get for T_1 =50s, T_2 =0, $\tau_1 = 0$, $\tau_2 = 0$, $\tau_3 = 0$

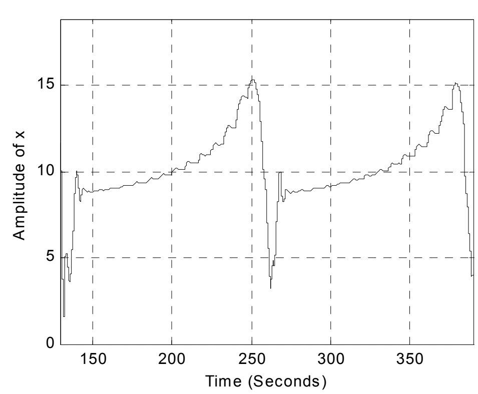

Fig.6 Variable's x oscillations get for T_1 =50s, T_2 =100s, $\tau_1 = 0$, $\tau_2 = 0$, $\tau_3 = 0$

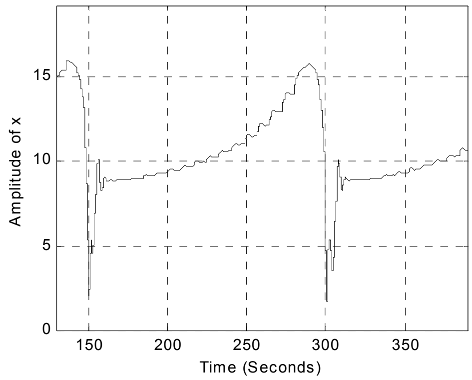

Fig.7 Variable's x oscillations get for
T_1 =50s, T_2 =100s, τ_1 = 20s, τ_2 = 0, τ_3 = 0

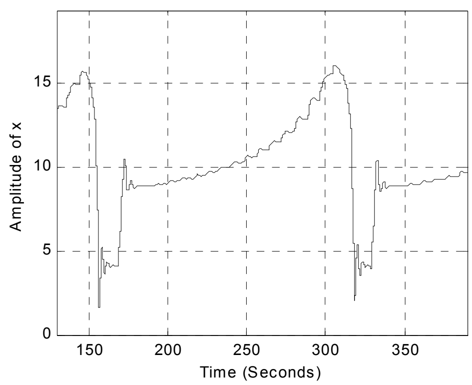

Fig.8 Variable's x oscillations get for
T_1 =50s, T_2 =100s, τ_1 = 20s, τ_2 =10s, τ_3 = 15s

oscillation's appearance are determined.
The simulation results obtained by numerical simulation confirm the theoretical considerations.

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