STABILITY ANALYSIS OF CELLULAR NEURAL NETWORKS WITH TIME DELAYS

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Abstract: Cellular Neural Networks (CNN) are artificial neural networks displaying multidimensional arrays of cells and local interconnections among the cells. Since in a CNN all the cells are identically, the qualitative behavior of the entire network can be studied via stability results obtained for the interconnected systems. The finite switching speed of amplifiers and communications time between cells of the VLSI technology CNN implementations introduce time lags that may lead to oscillations or to the instability of the network. The aim of the paper is to obtain sufficient conditions for the asymptotical stability of a cellular neural network displaying interaction delays. Within the framework of the qualitative theory of the large-scale composite systems these conditions are based on the properties of the Liapunov functions (functionals).

Keywords: cellular neural network, interconnected systems, large-scale composite systems, time delays, asymptotical stability.

1. INTRODUCTION

Cellular neural networks (CNN) are recurrent neural networks displaying a multidimensional array of cells and local interconnections among the cells. CNN have been successfully applied to signal processing, image processing, shape extraction and edge detection. Such applications depend mainly on the dynamical behavior of the network. Therefore, the analysis of these dynamical behaviors is a necessary step for the practical design of the neural network and has to be checked on the mathematical model.

Since in a CNN all the cells are identically, the qualitative behavior of the entire network can be studied via stability results obtained for the interconnected systems (Šiljak 1972, Willems 1972, Michel 1974, Vidyasagar 1981). The stability analysis of such systems can be accomplished in terms of the isolated subsystems and in terms of the interconnecting structure. Consequently, first one may state stability criteria for an isolated cell with their dynamics (results obtained by the author in a preview work (Danciu and Rasvan 2001) and stated here for the sake of the completeness), then one considers the interconnections, which are nonlinear, and imposes conditions for the asymptotical stability preservation.

2. THE MATEMATICAL MODEL AND PROBLEM STATEMENT

The aim of this paper is to obtain sufficient conditions for the asymptotical stability of a cellular neural network with time delay feedback and zero control templates

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j \in N} w_{ij} f\left(x_j(t - t_j)\right) + I_i \tag{1}$$

where j is the index for the cells of the nearest neighborhood N of the ith cell, a_i is a positive parameter and I_i is the bias.

The dynamics of the isolated cell is described by

$$\dot{x}_i = -a_i x_i(t) + w_{ii} f\left(x_i(t - t_i)\right) + I_i \tag{2}$$

The nonlinearities

$$f(x_i) = \tfrac{1}{2}\left(|x_i + 1| - |x_i - 1|\right) \tag{3}$$

are bounded, their range being [-1, 1]. Also these functions are monotonically increasing and globally Lipschitzian. This means they satisfy the inequalities

$$0 \le \frac{f(s_1) - f(s_2)}{s_1 - s_2} \le L \tag{4}$$

but also the sector condition

$$0 \le \frac{f(s)}{s} \le L \tag{5}$$

since $f(0) = 0$, where the Lipschitz constant is $L = 1$.

The equilibrium point x^* can be shifted to the origin, so that systems (1) and (2) can be written into the form:

$$\dot{z}_j(t) = -a_i z_i(t) + \sum_{j \in N} w_{ij} g\left(z_j(t - t_j)\right), \quad i = \overline{1,n} \tag{6}$$

$$\dot{z}_i = -a_i z_i(t) + w_{ii} g\left(z_i(t - t_i)\right) \tag{7}$$

where

$$z_i = x_i - x_i^* \text{ and } g(z_i) = f(z_i + x_i^*) - f(x_i^*).$$

Function $g(z)$ satisfies the same Lipschitz condition as $f(x)$ with the same constant $L = 1$. The properties of the nonlinear functions suggest application of the absolute stability theory methods. All approaches in the theory of large-scale composite systems are based on the properties of Liapunov functions (functionals).

3. MAIN RESULT

Theorem: *The equilibrium $x = 0$ of the system (6) is asymptotically stable if the following conditions are satisfied:*

i) $a_i > 0$;

ii) the nonlinear functions $g(s)$ are globally Lipschitz satisfying

$$0 \le \frac{g(s_1) - g(s_2)}{s_1 - s_2} \le 1, \quad g(0) = 0$$

iii) for each isolated subsystem (7) there exists a Liapunov functional $V_i : \mathbf{R} \times \mathbf{L}^2(-t_i, 0; \mathbf{R})$ and the parameters $a_i > 0$ and $b_i > 0$ such that $b_i < a_i(a_i + 1)$ and $|w_{ii}| < a_i$;

iv) there exist the constants $g_i > 0$, $i = \overline{1,n}$ such that the symmetric matrix $B = (b_{ij})_{n \times n}$ definited by

$$b_{ii} = g_i b_i - \frac{1}{4} \sum_{j=1}^{n} |w_{ji}|^2 \frac{g_j (a_j + 1)^2}{a_j(a_j + 1) - b_j}$$

$$b_{ij} = -\frac{1}{4} \sum_{j=1}^{n} |w_{ki}| |w_{kj}| \frac{g_k (a_k + 1)^2}{a_k(a_k + 1) - b_k} = b_{ji}$$

to be positive definite.

Following the procedure sketched in the introduction of the paper, one considers first an isolated cell with its dynamics described by (7) and one chooses the following Liapunov functional candidate defined e.g. on Hilbert space $\mathbf{R} \times \mathbf{L}^2(-t_i, 0; \mathbf{R})$:

$$V_i(z, j) = \tfrac{1}{2} a_i z^2 + b_i \int_{-t_i}^{0} j^2(q)\,dq + \int_0^z g(s)\,ds \tag{8}$$

where $a_i > 0$, $b_i > 0$ are suitably chosen arbitrary parameters. Along the solutions of (7) this functional reads

$$V_i^*(t) = V_i(z_i(t), z_{it}) = \tfrac{1}{2} a_i z_i^2(t) + \\ + b_i \int_{t-t_i}^{t} z_i^2(q)\,dq + \int_0^{z_i(t)} g(s)\,ds \tag{9}$$

Taking into account the nonlinearities, which satisfy Lipschitz conditions (5) with the constant $L = 1$, from the negative definiteness of the derivative of $V_i^*(t)$ along the solutions of (7) one obtains (see [1] for detailes) the "small gain conditions" $|w_{ii}| < a_i$ and the following choice for a_i and b_i:

$$b_i < a_i(a_i + 1) \tag{10}$$

$$\frac{1}{2}(a_i + 1)\left[1 - \sqrt{1 - \left(\frac{w_{ii}}{a_i}\right)^2}\right] < \frac{b_i}{a_i} < \\ < \frac{1}{2}(a_i + 1)\left[1 + \sqrt{1 - \left(\frac{w_{ii}}{a_i}\right)^2}\right]. \tag{11}$$

The role of the interconnections structure is very important for the asymptotical stability preservation. In the sequel, we consider the dynamic system represented by a cellular neural network and described by functional differential equations (6). This composite system may be viewed as a nonlinear interconnection of n isolated subsystems described by (7):

$$\dot{z}_i = -a_i z_i(t) + w_{ii} g\left(z_i(t - t_i)\right) + \\ + \sum_{\substack{j=1 \\ j \ne i}}^{n} w_{ij} g\left(z_j(t - t_j)\right), \quad i = \overline{1,n} \tag{12}$$

For this system we choose the following Liapunov functional candidate

$$V(z, j) = \sum_{i=1}^{n} g_i V_i(z, j), \quad g_i > 0, \quad i = \overline{1,n} \tag{13}$$

with $V_i(z, j)$ from (8). Making use of (9), the

functional (13) along the solutions of system (6) becomes:

$$V^*(t) = V(z(t), z_t) =$$
$$\sum_{i=1}^{n} g_i \left[\tfrac{1}{2} a_i z_i^2(t) + b_i \int_{t-t_i}^{t} z_i^2(q)dq + \int_0^{z_i(t)} g(s)ds \right], \quad (14)$$

$$g_i > 0, \quad i = \overline{1,n}$$

By reordering and using for the nonlinearities the Lipschitz conditions (5) with the constant $L = 1$, we obtain the following estimate for the derivative of Liapunov functional (14) along the solutions of the system (6):

$$\frac{dV^*}{dt}(t) \le -\left\{ \sum_{i=1}^{n} g_i \left(a_i(a_i+1) - b_i\right) \left| z_i(t) \right|^2 + \right.$$
$$+ \sum_{i=1}^{n} g_i b_i \left| z_i(t-t_i) \right|^2 - \quad (15)$$
$$\left. - \sum_{i=1}^{n} g_i (a_i+1) \left| z_i(t) \right| \sum_{j=1}^{n} \left| w_{ij} \right| \left| z_j(t-t_j) \right| \right\}$$

By denoting

$$y(t) = [z_1(t), z_2(t), \cdots, z_n(t), z_1(t-t_1), z_2(t-t_2), \cdots, z_n(t-t_n)]^T \quad (16)$$

$$\left| W \right| = \left[\left| w_{ij} \right| \right]_{n \times n} \quad (17)$$

$$M = \begin{bmatrix} diag\left(g_i[a_i(a_i+1) - b_i]\right)_1^n & -\tfrac{1}{2} diag\left(g_i(a_i+1)\right)_1^n \left| W \right| \\ -\tfrac{1}{2} \left| W^T \right| diag\left(g_i(a_i+1)\right)_1^n & diag\left(g_i b_i\right)_1^n \end{bmatrix} \quad (18)$$

it is obvious that the derivative of $V^*(t)$ is negative definite if the matrix M of the quadratic form $y^T(t) M\, y(t)$ with $2n$ variables is positive definite. Since $a_i, b_i, g_i, a_i > 0$ first condition

$$\Delta_1 = \det\left(diag\left(g_i[a_i(a_i+1) - b_i]\right)_1^n \right) > 0 \quad (19)$$

is verified for $b_i < a_i(a_i+1)$, $i = \overline{1,n}$. By rewritting the determinant of matrix M using Schur formula, the second condition

$$\Delta_2 = \det M > 0 \quad (20)$$

impose the symmetric matrix

$$B = diag\left(g_i b_i\right)_1^n - \tfrac{1}{4} \left| W^T \right| diag\left(\frac{g_i(a_i+1)^2}{a_i(a_i+1) - b_i} \right)_1^n \left| W \right| \quad (21)$$

to be positive definite.

4. CONCLUDING REMARKS

The present paper extend our preview work (Danciu and Rasvan 2001), and state sufficient condition for the asymptotical stability of a cellular neural network with time delay feedback and zero control templates. Since CNN consist of identical cells, the analysis is made within the framework of the qualitative theory of the large-scale composite systems. The result is based mainly on the properties of the Liapunov functionals and is independent of the delays.

5. REFERENCES

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