Youla-Kučera parameter in explicit control laws

Pedro Rodriguez-Ayerbe * **Sorin Olaru** *

* *Automatic Control Department of SUPELEC (E3S), 91192 Gif-sur-Yvette, France (Tel: +33-143267485; e-mail: pedro.rodriguez@supelec.fr, sorin.olaru@supelec.fr).*

Abstract: The paper deals with the predictive control for linear systems, described by piecewise affine (PWA) control laws formulations. The main goal is to reduce the sensitivity of these schemes with respect to the model uncertainties. This objective can be attained by considering worst-case (min-max) formulations, optimization over the control policies or tube predictive control. These comprehensive approaches may lead to fastidious on-line optimization thus reducing the range of application. In the present paper, a two stage predictive strategy is proposed, which synthesizes in the first place an analytical (continuous and piecewise linear) control law based on the nominal model and secondly robustifies the control law in the neighborhood of the equilibrium point using the Youla-Kučera parameter (the feedback gain obtained for the unconstrained control problem - most often assimilated to the LQR gain). This robustification is globally expanded to all the state space of the piecewise structure by means of its corresponding disturbance model.

1. INTRODUCTION

The model predictive control (MPC) laws are optimisation based techniques which allow constraints handling from the design stage. Their practical implementation is related to the real-time computation of a finite horizon optimal control sequence. The analytical formulation of the optimum and its on-line evaluation avoids the important computational effort required for real-time optimisation. Solutions in this direction exist for linear and quadratic cost functions subject to linear constraints thanks to the Abadie constraint qualification (Goodwin et al. [2004]). It must be said that these are in fact a part of a larger class of parametric convex programs, see Pistikopoulos et al. [2007], for which exact or approximate algorithms exist, see Bemporad et al. [2002b], Olaru and Dumur [2004], Seron et al. [2003], Grancharova et al. [2007], Bemporad and Filippi [2006].

In the case of robust predictive control laws, the model uncertainties and the disturbances can be taken into account at the design stage. A popular methodology in this direction is the one based on a min-max criterium (when the extreme combination of disturbances or uncertainties are known), see e.g. Kerrigan and Maciejowski [2004], Bemporad et al. [2002a], Olaru and Dumur [2007], which comes finally to the resolution of a single parametric linear program. The structure of this ultimate optimisation is however quite complex and large prediction horizons cannot be handled due to the exponential growth of disturbance realisations that have to be taken into account. The exact explicit solutions being prohibitive in terms of computational complexity, Grancharova and Johansen [2009] proposed as an interesting alternative the construction of approximations. Different approaches emerged in the last decade for an optimisation over the control policies instead of an optimisation over the control actions, thus leading to attractive robust formulation, see Løvaas et al. [2008], Goulart et al. [2006]. Tube MPC in Langson et al. [2004] is another approach to this complex robust control problem and is somehow connected to the output feedback MPC studies of Mayne et al. [2006]. Furthermore, we note here the fact that the input to state stability concepts were adapted to robust MPC context in the recent studies see e.g. Lazar [2006], Limon et al. [2008], with implications to the systems/control law presenting discontinuities. We have thus the picture of a growing interest for the robustness issues related to the MPC synthesis.

In the present paper we will approach this problem in a slightly different manner, close to the construction of an estimation mechanism for the constrained variables. In Goodwin et al. [2004], a robust control structure is obtained but the parametric optimisation remains intricate as long as the feasible domains are not polytopic. A first study regarding the possible robustness improvement for the explicit affine feedback policy constructed upon predictive control strategy for linear systems was presented in Olaru and Rodriguez-Ayerbe [2006]. The simplest way to proceed is to consider an observer of the state variables as in Goodwin et al. [2004]. The use of an observer preserves the dimension of the state space and by consequence the piecewise structure of controller. An interesting feature is the fact that the same observer can be used over the entire domain independently of the active region of the controller. We note that the observer can also be considered as a noise characterization for the prediction model. Nevertheless, the observer does not allow spanning the entire space of stabilizing controllers.

The present paper introduces an improved result based on the Youla-Kučera parametrization which spans the space of stabilizing controllers. For a two-degree of freedom controller, one has access to all the stabilizing controllers that preserve the same tracking behavior, so the Youla-Kučera parameter offers more degrees of freedom than the use of an observer. The robustification is made such that the state space dimension of the controller is augmented. The direct consequence is that the use of the same parameter in each region is not possible. The continuity between critical regions can be lost with severe degradation in stability and performances. The main contribution here is the reconstruction of the noise model induced by the Youla-Kučera parameter for the unconstrained case, and its use for the generation of the robust piecewise controller corresponding to the constrained MPC case.

In the following, section 2 briefly recalls the constrained MPC control, and the explicit solution to the associated parametric optimisation problem. Section 3 considers the robustification of a linear controller using the Youla-Kučera parameter and the equivalent disturbance model. Numerical examples are presented in section 4 and the final conclusions is drawn in section 5.

2. CONSTRAINED MPC

2.1 From receding horizon control problem to the QP/LP formulation

The design of a predictive control law is based on the existence of an analytic/simulation model of the system to be controlled. In the linear time invariant framework, consider the state space model:

$$\begin{array}{l} x_{t+1} = Ax_t + Bu_t \quad t \in Z^+ \\ y_t = Cx_t + Du_t \end{array} \tag{1}$$

With $x_t,\ x_{t+1} \in \mathbb{R}^n$ the state vector at time t and $t + T_e$ respectively, $u_t \in \mathbb{R}^m$ the control vector at time t; A and B matrices of adequate dimensions and the pair (A, B) assumed to be stabilisable.

At each sampling time, the current state vector (assumed to be measurable) $x_t = x_{t|t}$ is used to elaborate the open loop optimal control sequence $\mathbf{u}^*$:

$$\mathbf{u}_t^* = \left[\, u'_{t|t} \ \cdots \ u'_{t+N-1|t} \,\right]' \tag{2}$$

with respect to a given cost function:

$$\mathbf{u}_t^* = arg \min_{\mathbf{u}_t} \|Px_{t+N|t}\|_p + \sum_{k=0}^{N-1} \{\|Qx_{t+k|t}\|_p + \|Ru_{t+k|t}\|_p\} \tag{3}$$

where $\|.\|_p$ represents the norm $p = \{1, 2, \infty\}$ and the pair (Q, A) is assumed to be detectable. The prediction horizon N, the weighting terms $Q = Q' \geq 0$, $R = R' > 0$ and the final cost defined by $P = P' \geq 0$ are the tuning knobs of the control law.

The optimisation of this cost function is performed subject to constraints imposed by the system dynamic, the functional constraints and terminal or stability constraints:

$$\begin{cases} x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t} \quad k \geq 0 \\ H_x x_{t+k|t} + H_u u_{t+k|t} \leq \gamma,\ 0 \leq k \leq N, \\ x_{t+N|t} \in X_N \end{cases} \tag{4}$$

It is considered in the following that all constraints in (4) are of polyhedral type. The finite set of constraints can be restructured to obtain a compact formulation:

- Case p = 2:

$$\begin{array}{l} \mathbf{u}_t^* = arg \min\limits_{\mathbf{u}_t} 0.5\mathbf{u}_t' H\mathbf{u}_t + x'F\mathbf{u}_t \\ \texttt{subjet to}: \quad G\mathbf{u}_t \leq W + Sx \end{array} \tag{5}$$

- Case p = 1,∞:

$$\begin{array}{l} \mathbf{z}^* = arg \min\limits_{\mathbf{z}} c^T\mathbf{z} \\ \texttt{subjet to}: \quad G\mathbf{z} \leq W + Sx \end{array} \tag{6}$$

  with $\mathbf{z} = \{\mathbf{u}_t; \xi_1, \ldots, \xi_{N_\xi}\}$ and $\xi_1, \ldots, \xi_{N_\xi}$ auxiliary variables, the number N_ξ of these variables depending on the optimisation horizon and the prediction model Zadeh and Whalen [1962].

For both cases (5) - (6), the optimal argument includes the control sequence $\mathbf{u}_t^*$. Only the first part of this sequence is applied effectively to the system input, the complete procedure is reiterated at the next sampling time according to the receding horizon principle, see e.g. Mayne et al. [2000]. Real time implementation is usually performed through on-line optimisation procedures (linear or quadratic programming) in order to determine the optimum corresponding to a particular value of the state vector x.

In the following section we concentrate on the explicit formulations for the predictive control law. We focus on the quadratic case by exploiting the uniqueness and continuity of the solution in this case. One should note that the same results can be obtained for the LP formulations as long as a continuous selection is assured for the optimal solution see e.g. Olaru and Dumur [2006], Spjøtvold et al. [2007].

2.2 Explicit solution for quadratic case

The analytic solution of (5) - (6) can be constructed along the lines of sensitivity analysis for parametric optimisation problems (see Pistikopoulos et al. [2007] for a review of the control problems under these framework). The optimal solution will be expressed as an explicit function of the state vector x.

$$f : \mathbb{R}^n \to \mathbb{R}^m \ \texttt{so that} \ u_t^{MPC} = f(x_t) \tag{7}$$

For the quadratic cost functions and the analytical solution of the parametric Quadratic Program (8), see for example the review paper Alessio and Bemporad [2008].

$$\textit{QP(x)}: \quad \begin{array}{l} V^*(x) = \dfrac{1}{2}x'Yx + \min\limits_{\mathbf{z}} \dfrac{1}{2}\mathbf{u}_t' H\mathbf{u}_t + x'F'\mathbf{u}_t \\ \qquad \texttt{s.t.} \quad G\mathbf{u}_t \leq W + Sx \end{array} \tag{8}$$

Several studies were dedicated to the geometry of the piecewise affine characterization see e.g. Bemporad et al. [2002b], Seron et al. [2003], Olaru and Dumur [2004], Mare and DeDona [2005]. Real time implementation is reduced in this case to the evaluation of this function.

Regarding the structure of the multiparametric problems it can be observed that the feasible domain is represented by a parameterized polyhedron. If bounded, then the optimum is given by a convex combination of parameterized vertices. If the optimal solution is not unique (usually the case of linear cost functions (6)), the explicit solution is equivalent to a point to set mapping as showed in Olaru and Dumur [2006], and the continuity of the solution must be a crucial criterion when implementing the solution. Indeed, a continuous control law avoids discontinuous variations on the control in case of disturbances appearing on the state vector.

The use of a dual representation of the feasible domain and projection mechanisms (see Olaru and Dumur [2004] - Olaru and Dumur [2005] for details) provides an insight on the topology of the optimisation problems and can be advantageous if there exist unbounded directions due to the fact that the generators representation offers the right tool for their description as well as for the control of the constraints redundancy. Once the explicit solution of (5) - (6) is obtained, we dispose of an

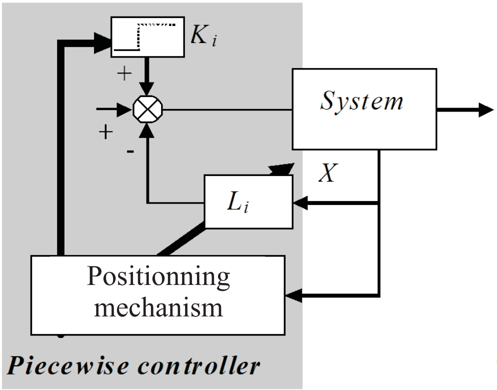

Fig. 1. Piecewise affine controller implementation

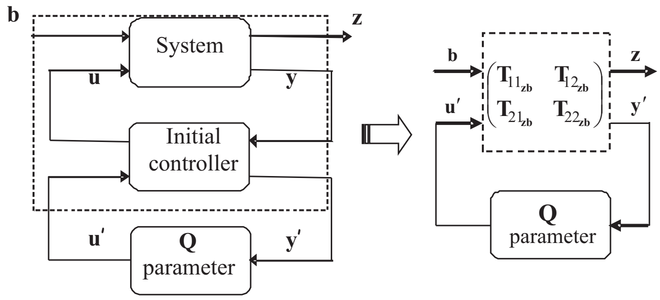

Fig. 2. Class of all stabilizing controllers

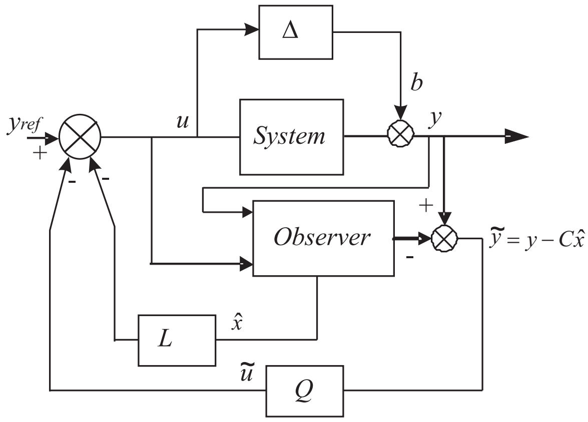

Fig. 3. Robustified controller via the **Q** parameter

analytic description of the control law. Several studies were dedicated to the piecewise affine characterisation (Bemporad et al. [2002b], Seron et al. [2003], Olaru and Dumur [2004], Mare and DeDona [2005]).

Indeed, the explicit predictive control law is described by a collection of piecewise affine function:

$$u_t^{MPC} = f(x_t) = \begin{cases} L_1 x_t + l_1 & if \;\; x_t \in R_1 \\ \cdots & \cdots \\ L_k x_t + l_k & if \;\; x_t \in R_k \\ \cdots & \cdots \end{cases} \tag{9}$$

with R_k polyhedral critical regions covering feasible states.

The structure of such a piecewise controller is shown in Fig. 1. Once the look-up table of local laws is available, an efficient positioning mechanism, as the one proposed in Tøndel et al. [2003] based on a search tree, can be constructed such that the on-line evaluation routine can find the optimal control action.

The implementation of the controller is based on the availability of the current state. For the case when the state is not directly measured, optimal solution is proposed in Perez et al. [2004]. Nevertheless, the optimality in this case does not usually justify the complexity of the solution and thus the use of an observer is usually considered in Goodwin et al. [2004].

3. YOULA-KUČERA PARAMETRIZTION AND NOISE MODEL

3.1 Generalities

This paragraphe proposes the obtention of the disturbance model through the synthesis of a Youla-Kučera parameter, see e.g. Boyd and Barratt [1991], Kouvaritakis et al. [1992], Rodríguez and Dumur [2005], Rossiter [2003], robustifying the central controller (corresponding to an empty subset of active constraints in the MPC formulation).

The Youla-Kučera parametrization is a well known technique in the literature, and its main advantage is to provide a representation of an entire class of stabilizing controllers. In fact, the Youla parameter, denoted **Q** parameter, establishes a bijection between the class of all stable transfer functions and the class of all stabilizing controllers. If it is inserted in a special way in a closed-loop, the **Q** parameter does not affect the tracking behavior, but it allows modifying the sensitivity functions in order to improve robustness of the controlled system Rossiter [2003].

Known as the modified controller paradigm Boyd and Barratt [1991], the **Q** parametrization begins with the addition of supplementary inputs and outputs (into the controller) with a zero transfer between them ($T_{22_{zb}} = 0$ in Fig. 2), which enables to connect a stable Youla parameter between z and b without restricting the closed-loop stability. As a result, the closed-loop function between b and z depends in an affine way on the **Q** parameter, allowing convex specifications:

$$T_{zb} = T_{11_{zb}} + T_{12_{zb}} \mathbf{Q}\, T_{21_{zb}} \tag{10}$$

Considering a system (1) with a state feedback controller and an observer (11), the **Q** parameter is added as shown in Fig. 3. L is a static feedback gain, and **Q** is a dynamical system. Applying the small gain theorem, robustification of this structure towards unstructured uncertainties, as the additive one shown in Fig. 3, can be performed by minimization of H_∞ norm of the transfer seen by the uncertainty. This problem is convex and can be solved by linear programming or LMI, see Rodríguez and Dumur [2005], Stoica et al. [2007].

$$\begin{aligned} \hat{x}_{t+1} &= A\hat{x}_t + Bu_t + K(y_t - C\hat{x}_t) \\ u_t &= -L\hat{x} - \tilde{u}_t \end{aligned} \tag{11}$$

In the case of the uncertainty shown in Fig. 3, the robustification can be expressed as:

$$\min_{\mathbf{Q} \in \mathbb{R}H_\infty} \|WT_{ub}\|_\infty \tag{12}$$

where $\mathbb{R}H_\infty$ is the space of stable transfers, W is a weighting transfer function that reflects the frequency ranges where model uncertainties are more important, and T_{bu} represents the transfer between b and u in Fig. 3.

Other class of specifications, as for example temporal templates, can be used for the synthesis of the **Q** parameter, see e.g. Rossiter [2003], Rodríguez and Dumur [2005], Stoica et al. [2007] for details.

3.2 Disturbance model of Q parameter

Considering a state space (A_Q, B_Q, C_Q, D_Q) representation for the **Q** parameter:

$$\begin{cases} x_{Q_{t+1}} = A_Q x_{Q_t} + B_Q \tilde{y}_t \\ \tilde{u}_t = C_Q x_{Q_t} + D_Q \tilde{y}_t \\ \tilde{y}_t = y_t - C\hat{x}_t \end{cases} \tag{13}$$

The controller obtained with a state feedback gain L, an observer (11) and a **Q** parameter is:

$$\begin{aligned} \begin{bmatrix} \hat{x}_{t+1} \\ x_{Q_{t+1}} \end{bmatrix} &= A_{c_{QL}} \begin{bmatrix} \hat{x}_t \\ x_{Q_t} \end{bmatrix} + B_{c_{QL}} y_t \\ u_t &= C_{c_{QL}} \begin{bmatrix} \hat{x}_t \\ x_{Q_t} \end{bmatrix} - D_Q y_t \end{aligned} \tag{14}$$

and:

$$\begin{aligned} A_{c_{QL}} &= \begin{bmatrix} A - KC - BL + BD_QC & -BC_Q \\ -B_QC & A_Q \end{bmatrix} \\ B_{c_{QL}} &= \begin{bmatrix} K - BD_Q \\ B_Q \end{bmatrix} C_{c_{QL}} = \begin{bmatrix} -L + D_QC & -C_Q \end{bmatrix} \end{aligned} \tag{15}$$

In order to obtain the disturbance model corresponding to the **Q** parameter, we consider an augmented model of (1). We consider that the unknown dynamics and uncertainties in the model are represented by a noise model. Considering the innovation representation of Åström and Wittenmark [1997] we obtain:

$$\begin{cases} x_{t+1} = Ax_t + Bu_t + Ke_t \\ y_t = Cx_t + e_t \end{cases} \tag{16}$$

e_t representing a filtered white noise:

$$\begin{cases} x_{v_{t+1}} = A_v x_{v_t} + B_v v_t \\ e_t = C_v x_{v_t} + v_t \end{cases} \tag{17}$$

with v_t a zero mean white noise. The extended model is:

$$\begin{aligned} x_{e_{t+1}} &= A_e x_{e_t} + B_e u_t + K_e v_t \\ y_t &= C_e x_{e_t} + v_t \end{aligned} \tag{18}$$

where:

$$\begin{aligned} x_e &= \begin{bmatrix} x \\ x_v \end{bmatrix} A_e = \begin{bmatrix} A & KC_v \\ 0 & A_v \end{bmatrix} \\ B_e &= \begin{bmatrix} B \\ 0 \end{bmatrix} K_e = \begin{bmatrix} K \\ B_v \end{bmatrix} C_e = \begin{bmatrix} C & C_v \end{bmatrix} \end{aligned} \tag{19}$$

The system is partially controllable, A_v describing non controllable but observable modes. The predictive control law can be reformulated upon this new prediction model by maintaining the same cost function and constraints. The new pQP is given by:

$$\begin{aligned} &QP_e(x)\text{: } V^*(x_e) = \frac{1}{2}x_e'Y_e x_e + \min_{\mathbf{u}} \frac{1}{2}\mathbf{u}'H_e\mathbf{u} + x_e'F_e'\mathbf{u} \\ &\texttt{s.t.} \quad G\mathbf{u} \le W + S_e x_e \end{aligned} \tag{20}$$

The different matrices and vectors of (20) can be decomposed in two parts; one dependent on the x, the controllable part, and a second one dependent on x_v, the non controllable part:

$$\begin{aligned} &F_e = [F \quad F_v] \\ &H_e = H \quad \texttt{because } A_v \texttt{ is non controllable} \\ &S_e = [S \quad S_v] \end{aligned} \tag{21}$$

With this decomposition, the solution of the pQP can be splited in two parts, one dependent on x another dependent on x_v. It must be noted that the solution dependent on x is the same as the one considered in (9). The optimum without constraints is $u_t = -H^{-1}Fx - H^{-1}F_vx_v = -Lx - L_vx_v = -L_ex_e$. Considering that x and x_v are not measured but observed, the following observer is used:

$$\begin{aligned} \hat{x}_{e_{t+1/t}} &= A_e\hat{x}_{e_{t/t-1}} + B_eu_t + K_1(y_t - C_e\hat{x}_{e_{t/t-1}}) \\ \hat{x}_{e_{t/t}} &= \hat{x}_{e_{t/t-1}} + K_2(y_t - C_e\hat{x}_{e_{t/t-1}}) \end{aligned} \tag{22}$$

We consider the general case of an estimator observer, as the case of one predictor observer is obtained for $K_2 = 0$. The control signal becomes $u_t = -L\hat{x} - L_v\hat{x}_v = -L_e\hat{x_e}$. The obtained controller takes the form:

$$\begin{aligned} \hat{x}_{e_{t+1/t}} &= A_{cn}\hat{x}_{e_t} + B_{cn}y_t \\ u_t &= C_{cn}\hat{x}_t - L_eK_2y_t \end{aligned} \tag{23}$$

and:

$$\begin{aligned} &A_{cn} = (A_e - K_1C_e - B_eL_e(I - K_2C_e)) \\ &B_{cn} = (K_1 - B_eL_eK_2) \quad C_{cn} = -(L_e(I - K_2C_e)) \end{aligned} \tag{24}$$

The idea thereafter is to find the disturbance model (A_v, B_c, C_v) in order to obtain the equivalence (14)≡(23). This equivalence is obtained by satisfying the following equations:

$$D_Q = L_eK_2 \tag{25}$$

$$C_{c_{QL}} = C_{cn} \tag{26}$$

$$B_{c_{QL}} = B_{cn} \tag{27}$$

$$A_{c_{QL}} = A_{cn} \tag{28}$$

From (26) we obtain $C_Q = L_v - D_QC_v$, from (27) $K_1 = \begin{bmatrix} K^T & B_Q^T \end{bmatrix}^T$, and from (28) $B_Q = B_v$ and $A_Q = A_v - B_vC_v$.

To resume: the disturbance model (A_v, B_v, C_v) corresponding to a state feedback L, a predictor observer gain K and a **Q** parameter (A_Q, B_Q, C_Q, D_Q) is given by $B_v = B_Q$ and the (A_v, C_v) solution of the following nonlinear equation system:

$$\begin{cases} A_Q - A_v + B_vC_v = 0 \\ C_Q - L_v + D_QC_v = 0 \\ L_v = H^{-1}F_v(A_v, C_v) \end{cases} \tag{29}$$

Using this principle in the construction of robustified controller one can note that the dependence $F_v(A_v, C_v)$ depends on the nature of the central controller. $F_e = [F \quad F_v]$ in (20) is constructed with A_e, B_e, C_e and thus depends non linearly on matrices (A_v, C_v).

The problem (29) can be solved using non linear optimisation techniques. It must be noted that given the nonlinear structure, the existence and uniqueness of (29) are not proved in the general case, but feasibility certificates can be obtained with classical optimization routines.

The dependence shown in (29) can be further detailed explicitly for controllers synthesized with infinite horizon and finite horizon. We show in the following the case for the infinite horizon.

3.3 Case of infinite horizon controller

As the considered cost function (3) has a terminal constraint, in the case of impose explicitly this terminal constraint, the unconstrained controller will correspond to the infinity horizon optimal controller. This controller can be obtained solving the following Riccati equation:

$$P = A_e'PA_e - (A_e'PB_e)(B_e'PB_e+R)^{-1}(A_e'PB_e)' + Q_r \tag{30}$$

The controller gain is:

$$L_e = (B_e'PB_e+R)^{-1}B_e'PA_e \tag{31}$$

The obtained controller, considering a predictor observer is:

$$\begin{aligned} \hat{x}_{e_{t+1}} &= (A_e - K_1C_e - B_eL_e)\hat{x}_{e_t} + K_1y_t \\ u_t &= -L_e\hat{x}_{e_t} \end{aligned} \tag{32}$$

The observer gain corresponds to K_e , as (19) is an innovation representation. The equivalence (14)≡(32) can be developed partitioning P, the solution of the Riccati equation (30) as the partition of A_e.

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{pmatrix} \tag{33}$$

Considering the weighting factor $Q_r = [C \quad C_v]'[C \quad C_v]$, in order to ponder the output of the system, equation (30) becomes:

$$\begin{aligned} P_{11} &= A'P_{11}A - (A'P_{11}B)(B'P_{11}B+R)^{-1}(A'P_{11}B)' + C'C \\ P_{12} &= (A-BL)'P_{12}A_v + (A_BL)'P_{11}KC_v + C'C \end{aligned} \tag{34}$$

And:

$$L_e = [\,L \;\; L_v\,] = (B'P_{11}B)^{-1}B'[P_{11}A \quad P_{12}A_v + P_{11}KC_v] \tag{35}$$

First equation of (34) corresponds to the Riccati equation of initial system, so L is the same as the one of the initial controller. Considering the equivalence (14)≡(32) we can remark, that in the case when a prediction observer is used, D_Q must be zero, because y_t is not used to estimate x_t . This imposes a structural constraint on the Q parameter. We can also remark, that with $D_Q = 0$ we can impose $B_v = B_Q$. After some developments, the following equations must be verified for the A_v, C_v, P_{12} matrices:

$$\begin{cases} A_Q = A_v - B_vC_v \\ C_Q = (B'P_{11}B+R)^{-1}B'(P_{12}A_v + P_{11}KC_v) \\ P_{12} = (A-BL)'P_{12}A_v + (A_BL)'P_{11}KC_v + C'C \end{cases} \tag{36}$$

The equations (36) can be practically solved using optimization techniques in order to obtain the unknowns A_v, C_v, P_{12}.

4. EXAMPLE

In order to fix the ideas a simple system with constraints on the control action is considered:

$$H(q^{-1}) = \frac{y_t}{u_t} = \frac{0.1q^{-1}}{1-0.9q^{-1}} \tag{37}$$

Adding an integral action for step disturbances rejection, the model becomes:

$$H(q^{-1}) = \frac{y_t}{\Delta u_t} = \frac{0.1q^{-1}}{(1-q^{-1})(1-0.9q^{-1})} \tag{38}$$

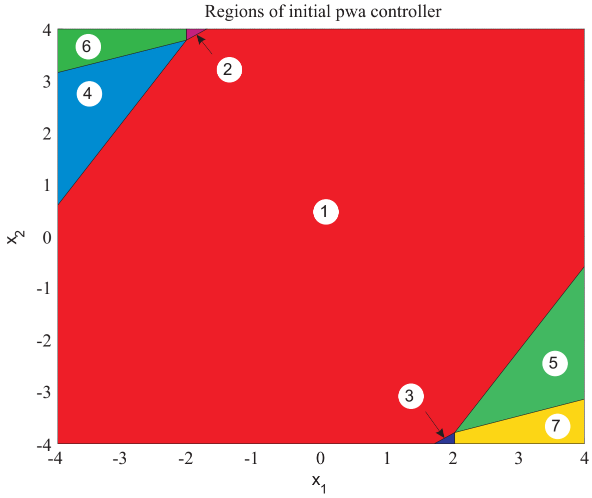

Fig. 4. Regions of initial PWA controller

The state space representation of the model retained is:

$$\begin{aligned} \begin{bmatrix} x_{1_{t+1}} \\ x_{2_{t+1}} \end{bmatrix} &= \begin{bmatrix} 0.9 & 0.1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1_t} \\ x_{2_t} \end{bmatrix} + \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} \Delta u_t \\ y_t &= [\,1 \;\; 0\,] \begin{bmatrix} x_{1_t} \\ x_{2_t} \end{bmatrix} \end{aligned} \tag{39}$$

We consider the following cost function

$$J = \sum_{k=N_1}^{N_2} (x_k'Qx_k) + \sum_{k=1}^{N_u} (\Delta u_k' R \Delta u_k) \quad Q \geq 0 \quad R > 0 \tag{40}$$

With $N_1 = 1, N_2 = 5, N_u = 2, R = 1, Q = C'C$. We take this Q in order to consider the output in the criteria. The constraints are:

$$|u| < u_{max} = 4 \quad |x_1| < 4 \quad |x_2| < u_{max} \tag{41}$$

The matrices of the obtained mpQP problem are:

$$\begin{aligned} H &= 2\Gamma'Q_m\Gamma + R_m \\ F' &= 2\Theta'Q_m\Gamma \\ Y &= \Theta'Q_m\Theta \\ G &= \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \quad W = \begin{bmatrix} u_{max} \\ u_{max} \\ u_{max} \\ u_{max} \end{bmatrix} \quad S = \begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{42}$$

With:

$$\begin{aligned} \Theta &= \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^{N_2} \end{bmatrix} \quad \Gamma = \begin{bmatrix} B & 0 \\ AB & B \\ \vdots & \vdots \\ A^{N_2-1}B & A^{N_2-2}B \end{bmatrix} \\ Q_m &= diag(Q,Q,Q,Q,Q) \quad R_m = diag(R,R) \end{aligned} \tag{43}$$

The obtained pQP problem has been solved using MPT Toolbox for Matlab (Kvasnica et al. [2006]). The obtained PWA controller is shown in Fig. 4. A controller of 7 regions is obtained. Each region and the obtained controller is summarized in table 1.

If the state is not measured, an observer is considered. We have considered a predictor observer, that is (11) with K, in order

Table 1. Initial PWA controller

Region	L	l
1	$[0.5720 \quad 0.2484]$	0
2	$[0 \quad 1]$	4
3	$[0 \quad 1]$	-4
4	$[0.1516 \quad 0.5084]$	1.8351
5	$[0.1516 \quad 0.5084]$	-1.8351
6	$[0 \quad 1]$	4
7	$[0 \quad 1]$	-4

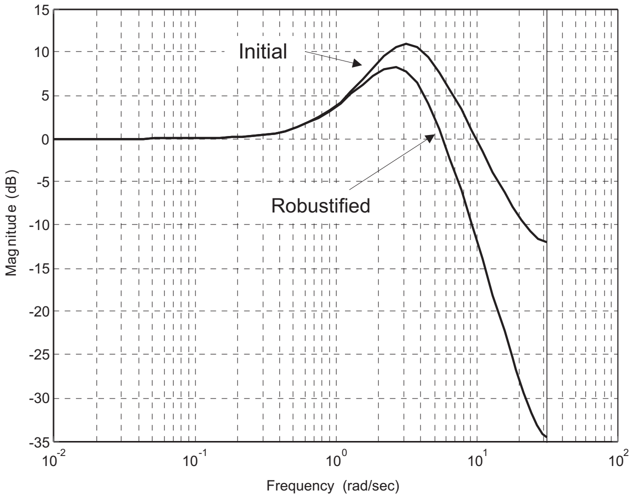

Fig. 5. T_{bu} transfer of initial and robustified central controllers

to place the poles of the observer in $[0.6 \quad 0.7]$. These poles have been chosen in order to have an observer faster than the closed loop with total information. The corresponding transfer T_{bu} is shown in Fig. 5. This transfer represents the sensitivity of the controlled system towards the additive unstructured uncertainties. The lower this transfer is, the bigger the accepted uncertainty without lost of stability is.

The robustification of the obtained central controller towards unstructured uncertainties gives $\mathbf{Q} = \frac{-0.5191q^{-1}}{1-0.8q^{-1}}$. That is $A_Q = 0.8, B_Q = 1, C_Q = 0.5191, D_Q = 0$. We have found a $\mathbf{Q}$ parameter with $D_Q = 0$, in order to keep a predictor observer, and of degree 1 in order to have an easier visualization of regions in the example. The T_{bu} transfer obtained with this $\mathbf{Q}$ parameter is shown in Fig. 5. As it can be observed, the robustification towards additive unstructured uncertainties is improved. The disturbance model corresponding to this $\mathbf{Q}$ parameter is obtained solving the following optimisation problem:

$$\begin{array}{l} A_Q - A_v + B_v C_v = 0 \\ C_Q - L_v + D_Q C_v = 0 \\ L_v = H^{-1} F_v(A_v, C_v) \end{array} \tag{44}$$

$F_v(A_v, C_v)$, dependent on A_v and C_v, is obtained as follows:

$$\begin{array}{l} F_e' = 2\Theta_e' Q_{e_m} \Gamma_e = [F' \quad F_v'] \\ Q_{e_m} = diag(Q_e, Q_e, Q_e, Q_e, Q_e) \\ Q_e = C_e' C_e \quad C_e = [C \quad C_v] \end{array} \tag{45}$$

Θ_e' and Γ_e are obtained with (43) using A_e and B_e of (19).

The solution of this non linear programming gives $A_v = -0.520, B_v = B_Q = 1, C_v = -1.3208$. The initial PWA controller can be modified according to this disturbance model. The

Table 2. Robustified PWA controller

Region	$L_e = [L \quad L_v]$	l
1	$[L_1 \quad -0.5169]$	l_1
2	$[L_2 \quad 0]$	l_2
3	$[L_3 \quad 0]$	l_3
4	$[L_4 \quad -0.8375]$	l_4
5	$[L_5 \quad -0.8375]$	l_5
6	$[L_6 \quad 0]$	l_6
7	$[L_7 \quad 0]$	l_7
8	$[0 \quad 1 \quad 0]$	4
9	$[0 \quad 1 \quad 0]$	-4

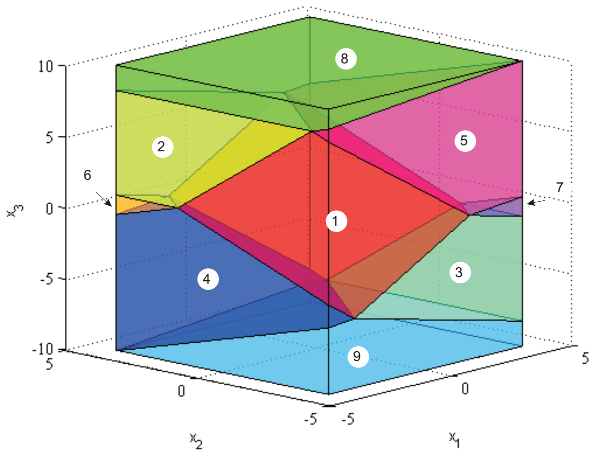

Fig. 6. Regions of robustified PWA controller

obtained PWA controller is shown in Fig. 6 and is summarized in table 2.

The obtained PWA controller has two more regions than the initial controller. In Fig. 6 it can be observed that, if the noise state is bigger than 5, two other regions are reached. This noise state and the state of the system is estimated by an observer (20) with $K_1 = [K \; B_v]$ and $K_2 = 0$ because $D_Q = 0$.

In order to show the robustification effect of the the Youla parameter, we consider simulation results obtained with a simulation model including a neglected high frequency dynamic. This dynamic corresponds to a second order system with $\omega_0 = 5$ and $\zeta = 0.08$. The Bode diagramme of simulation model and nominal model are shown in Fig. 7.

The considered neglected dynamic is high, but this is only to prove the pertinence of a very simple $\mathbf{Q}$ parameter (Fig. 5). Higher robustness can be obtained considering high order parameters. Simulation results are shown in Fig. 8 for both controllers. The figures show the output, the control signal and the active region. Fig. 8 shows the results in the case of using the observer with the poles in $[0.6 \quad 0.7]$, and Fig. 9 the results obtained with the robustified controller.

The results obtained with the observer are instable, as shown in Fig. 8. The use of a disturbance model can improve the results, as shown in Fig. 9, the system has no oscillations and is stable in closed loop.

5. CONCLUSIONS

The paper investigated the robustification methods for the control laws obtained in a constrained predictive control framework. The idea is to design in a first instance a piecewise

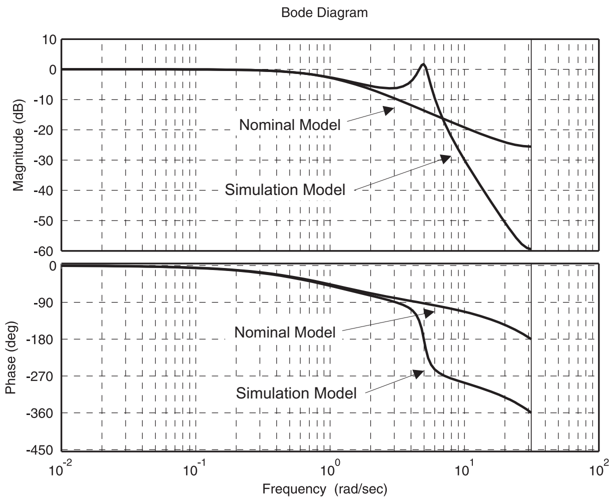

Fig. 7. Simulation and nominal model Bode diagramme

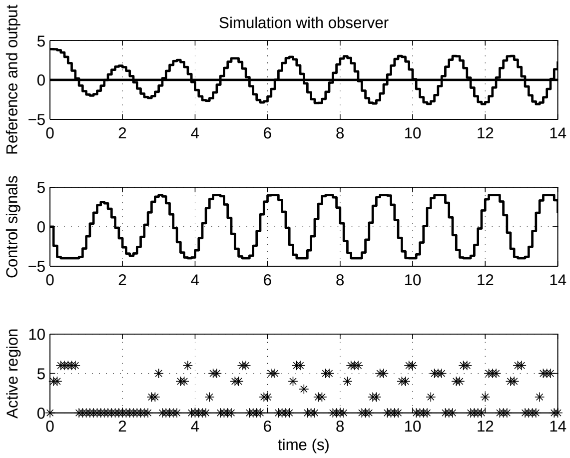

Fig. 8. Simulation results with initial observer

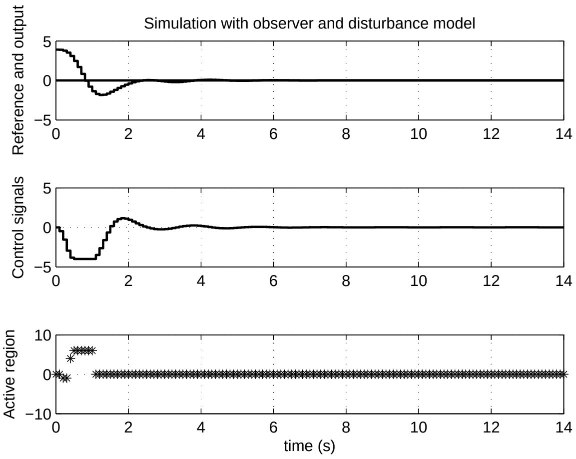

Fig. 9. Simulation results for robustified controller

controller which satisfies the basic demands in terms of tracking performances. In a second stage, the same predictive control structure (prediction horizon, weightings, etc.) is robustified using the model arguments accounting for the noise influence. Its has been shown that the structure of initial PWA controller is maintained. The robustified controller can be obtained from the initial one and the noise model parameters.

The robustification of initial unconstrained controller is made through the Youla-Kučera parametrization, and then this robustification is expanded to all the piecewise structure of the controller. For this, the disturbance model corresponding to the Youla-Kučera parameter is found, and used to regenerate the piecewise controller by preserving the same input/output behavior, but with an increased robustness.

The limitations of the method lay in the existence of the corresponding disturbance model of the Youla-Kučera parameter. This is transparent in the resolution of a non linear equation system. Given that the robustification is done off-line, any infeasibility can be handled by retuning the MPC parameters.

From another point of view, the approach can be seen as an extension of the robustification methods for linear systems to the control laws under constraints.

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