Monotone and slope restricted nonlinearities - a PIO II case study. ⋆

Vladimir Răsvan * **Dan Popescu** * **Daniela Danciu** *

* *Department of Automatic Control, University of Craiova, A.I.Cuza, 13 Craiova, RO-200585 Romania (e-mail: {vrasvan,dpopescu,daniela}@automation.ucv.ro).*

Abstract: In this paper there is presented a rather straightforward application of the absolute stability frequency domain inequalities to the practical problem of PIO (Pilot In-the-loop Oscillations) proneness of aircrafts. An extended to critical cases version of the Yakubovich criterion for the case of slope restricted nonlinearities is applied to the benchmark case of the X15 landing flare incident.

Keywords: Absolute stability, Frequency domain inequalities, Self sustained oscillations, Yakubovich criterion

1. PROBLEM STATEMENT AND STATE OF THE ART

This paper has two starting points.

A. The first one is theoretical and arises from the theory of absolute stability. A brief reminder, with reference to Fig. 1, *defines absolute stability as global asymptotic stability of the zero equilibrium of the system with the feedback structure there, this global asymptotic stability being valid for the entire class of systems defined (induced) by the class of nonlinear functions describing the nonlinear block of Fig. 1.*

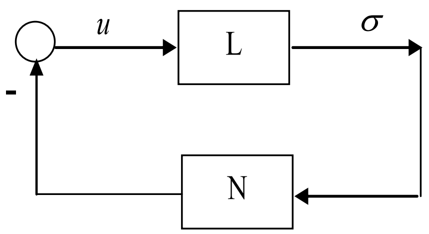

Fig. 1. Absolute stability feedback structure.

It should be added to this definition that the absolute stability conditions have to be expressed in the language of the linear sub-system and use information concerning the class of the nonlinearities (i.e. no specific nonlinearity of the class should be involved). An example at hand of these assertions is given below: if the nonlinear element is described by a function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ such that

$$0 \leq \varphi(\sigma)\sigma \leq \overline{\varphi}\sigma^2 \tag{1}$$

and the linear part by a strictly proper irreducible transfer function with its poles in $\mathbb{C}^-$, the Popov frequency domain inequality for absolute stability reads

$$\frac{1}{\overline{\varphi}} + Re(1 + \jmath\omega\theta)H(\jmath\omega) > 0, \;\; \forall\omega \in \mathbb{R}_+ \tag{2}$$

for some suitably chosen real θ.

As in most stability problems for nonlinear systems, the stability conditions are, generally speaking, only sufficient; the gap between them and the (possible) necessary and sufficient conditions, is called *method's "conservatism"*.

Aiming to reduce the "conservatism" suggested the researchers to impose additional restrictions on the nonlinear functions (by considering monotonic, odd monotonic, slope restricted etc nonlinearities). The general framework of this approach is summarized in the so-called Integral Quadratic Constraints (IQC) method as defined in the pioneering paper of Yakubovich Yakubovich (1967), developed, among others, in Răsvan (1975) and given the actual form in Megretski and Rantzer (1997). The idea is as follows: more restrictions are imposed on the nonlinear part of the system, less restrictive are the conditions on the linear part yielded by the frequency domain inequality or by the equivalent to it Liapunov function. In practice this means that the frequency domain inequality will contain more free parameters to choose in the necessary way, however this does not make the inequality easier to manipulate. Here also a trade-off appears as necessary: its significance is a necessary limitation of the number of IQCs that are considered in a specific problem and we shall deal with this fact in the following.

One of the oldest additional restrictions is the slope restriction deduced from the fact that the linear functions are both sector and slope restricted in the same sector. Starting with the suggestions of Kalman Kalman (1957), the results have been obtained by Yakubovich Yakubovich (1965a,b), extended to the case of several nonlinear elements in Barbălat and Halanay (1974) and to the case of hysteresis nonlinearities in Barabanov and Yakubovich (1979). All these papers take into account both sector and slope restrictions.

The note Singh (1984) aims to take into account the above mentioned trade-off by proposing "a stability criterion incorporating only the slope restrictions about the nonlinear function". The result was interesting but the proofs - far from convincing. In order to make the result credible several papers followed Răsvan (1988), Halanay and Răsvan (1991), Haddad and Kapila (1995), Haddad (1997), the reported criteria containing

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various restrictions concerning the linear part, the nonlinear part or the overall closed loop system. The presence of the restrictions witnessed about rigorous approaches but made more narrow the class of applications.

Consequently a comparison of all these approaches appears as both useful and necessary.

B. The second starting point is more application oriented. It is connected with PIO - P(ilot)-I(n the loop)-O(scillations), a rather complex phenomenon which can nevertheless be viewed as described by self-sustained oscillations of a feedback structure where the airframe dynamics represents the controlled object while the pilot dynamics acts for the controller. It is a well know fact now Anon. (2000) that there are three categories of PIO. They are defined as quasi-linear pilot-vehicle system oscillations except that series rate or position limiters are involved. At the physical level there are described by stating that "rate limiting", either as a series element or as a rate-limited surface actuator, modifies the Category I situation by adding an amplitude-dependent lag and by setting the limit cycle amplitude" Klyde et al. (1996).

The saturation is in fact the basic nonlinear function of a rate limiter which is described as in Fig. 2.

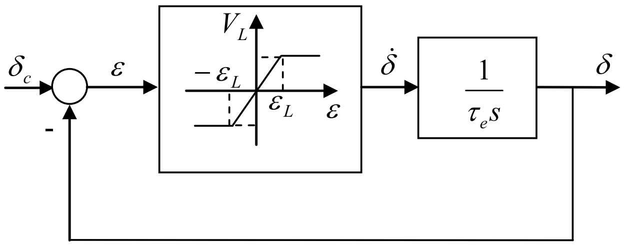

Fig. 2. Rate limiter

When incorporated in the PIO II structure the rate limiter will produce the diagram of Fig. 3 which at its turn may be reduced to the feedback loop of Fig. 1. The nonlinear block will be the saturation function described by

$$f(\varepsilon) = \begin{cases} V_L \mathrm{sgn}\,\varepsilon \;,\; |\varepsilon| \geq \varepsilon_L \\ \dfrac{V_L}{\varepsilon_L}\varepsilon \;,\; |\varepsilon| < \varepsilon_L \end{cases} \tag{3}$$

This nonlinear function is: sector restricted, monotone and slope restricted.

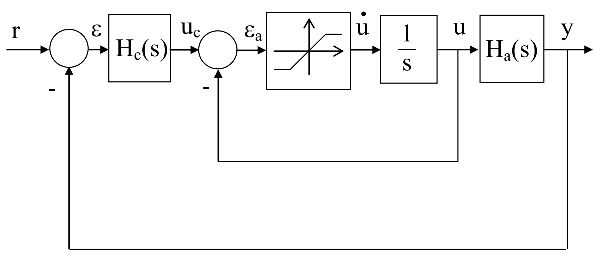

Fig. 3. Feedback structure with rate limiter

The resulting transfer function of the equivalent linear block will be, as expected

$$H(s) = \frac{1}{s}H_c(s)H_a(s) \tag{4}$$

The conclusion of this short exposition is that PIO II can be approached by absolute stability techniques.

C. We already gave some hints concerning the absolute stability criteria in the previous consideration. In a pioneering paper of Yakubovich Yakubovich (1967) it was shown that the properties of the nonlinear function may be expressed as some quadratic constraints (local or integral) on the input and output of the nonlinear block of Fig. 1; that paper is full of examples of association of such quadratic constraints. The approach was continued in Răsvan (1975) and become extremely popular in our days due to the contributions in Megretski and Rantzer (1997) where the quadratic constraints where considered for frequency domain also - see also Shiraev (2000). For our purposes it is important mentioning Megretski (1997) where the rate limiters are considered in the new context.

The achieved results correspond to the normal trends of the field of absolute stability: more information we have on the nonlinear subsystems i.e. more quadratic constraints are at our disposal, more free parameters are contained in the frequency domain inequality(ies). From here one obtains, again in a natural logic, that the sufficient conditions for stability thus obtained may be closer to the necessary and sufficient ones (less "conservative"). The counterpart is that more free parameters are, more difficult to manipulate (both analytically and numerically).

For these reasons the multi-parameter frequency domain stability inequalities are used with due caution: for instance, the Yakubovich type criterion for systems with monotonic nonlinearities Yakubovich (1965a,b) is easier to cope with than the Zames-Falb criteria Jonsson and Megretski (1997). This assertion will be tested on a PIO II model.

In order to end this introductory part we shall give here our methodology of applying the frequency domain inequalities. It is based on "parsimony principle" i.e. to have as few free parameters as possible in the frequency domain inequality to recover as much as possible from the linear stability sector for the nonlinearity one ("reducing the conservatism" i.e. the gap between sufficient and necessary and sufficient conditions for stability in the nonlinear case).

2. THE STABILITY INEQUALITIES FOR SLOPE RESTRICTED NONLINEARITIES

A. The oldest frequency domain inequality for sector and slope restricted nonlinearities appears in the papers of Yakubovich Yakubovich (1965a,b). If besides (1), the following slope restrictions are observed

$$\underline{\nu} < \varphi'(\sigma) < \overline{\nu} \tag{5}$$

and also the linear block is stable then the frequency domain inequality is

$$\begin{aligned} &\tau_1\left(\frac{1}{\overline{\varphi}} + \Re e H(\jmath\omega)\right) + \tau_2 \Re e\left(\jmath\omega H(\jmath\omega)\right) + \\ &+\tau_3\omega^2 \Re e\left(1+\underline{\nu}H(\jmath\omega)\right)^*\left(1+\overline{\nu}H(\jmath\omega)\right) \geq 0 \;,\; \forall\omega\in\mathbb{R} \end{aligned} \tag{6}$$

for some freely chosen $\tau_1 > 0$, $\tau_2 \in \mathbb{R}$, $\tau_3 \geq 0$.

For hysteresis-like nonlinearities the sign of τ_2 depends on some computed parameter ψ whose sign is determined by the sense on the hysteresis loop: it is required that $\tau_2\psi \leq 0$ Barabanov and Yakubovich (1979).

If only slope restrictions are taken into account, then we have to take $\tau_1 = 0$ (thus eliminating the possibility of S-procedure). In this way we are closer to the cases of Singh (1984), Haddad and Kapila (1995) or Halanay and Răsvan (1991). Requiring, additionally

$$(1 + \underline{\nu}H(0))(1 + \overline{\nu}H(0)) > 0 \tag{7}$$

we may write that (6) is equivalent to

$$\frac{\tau_3}{\nu_2} + Re\left[\left(\tau_3(1 + \underline{\nu}/\overline{\nu}) - \frac{\tau_2}{\overline{\nu}}\frac{1}{j\omega}\right)H(j\omega) + \right. \\ \left. +(\tau_3\underline{\nu})|H(j\omega)|^2\right] \geq 0 \tag{8}$$

and this is exactly the condition in Barabanov and Yakubovich (1979) or in Halanay and Răsvan (1991); note that in these papers the matrix A need not be Hurwitz but only hyperbolic.

It is thus obvious that the first frequency condition (6) is also the most general. Moreover, even the proof is performed in the least restrictive assumptions among all these methods.

B. Examine now conditions (6) or (8) which contain both the term $\underline{\nu}|H(j\omega)|^2$. Therefore, if $\underline{\nu} < 0$ this term is damaging the frequency domain inequality. But $\underline{\nu} \geq 0$ means that the nonlinear function is also monotonically increasing and, if $\overline{\nu} < +\infty$, globally Lipschitz. These properties are significant for the existence of forced oscillations. On the other hand monotonicity will allow introduction of the Zames-Falb or Brockett-Willems multipliers which might be helpful in certain frequency domain characteristics. We give below, for the sake of completeness, the Brockett-Willems criterion, reproduced after Popov (1973). Assume that besides (1) $\varphi(\sigma)$ is monotone increasing e.g. (5) holds for $\underline{\nu} = 0$ and $\overline{\nu}$ arbitrary; then the frequency domain inequality is as below

$$\frac{1}{\overline{\varphi}} + ReZ(j\omega)H(j\omega) \geq 0 \,,\ \forall\omega \in \mathbb{R} \tag{9}$$

where

$$Z(s) = 1 + s\theta + \sum_{1}^{p}\delta_j\frac{s + \rho_j}{s + \rho_j + \rho'_j} \tag{10}$$

where $\theta \geq 0$, $\delta_j > 0$, $\rho_j \geq 0$, $\rho'_j > 0$, $j = 1, ..., p$ are freely chosen parameters. Clearly we have here a typical example of a criterion with many free parameters and this does not make it easier to manipulate. The choice of all these parameters is connected with circuit synthesis since (9) is the expression of the condition of positive realness requirement which is basic in circuit synthesis; also $Z(s)$ may be viewed as a phase correction to ensure positive realness.

On the other hand the multiplier defined by (10) contains the non-causal part $1 + \theta s$ which is exactly the Popov multiplier; the additional term, being proper, is causal. The fact led other researchers to consider general non-causal multipliers: this is the case of the Zames-Falb multiplier Zames and Falb (1968), also of other ones Venkatesh (1970), Gheţaru (1969)

3. A CASE STUDY - THE X-15 LANDING FLARE PIO

This aviation incident occurred on June 8, 1959 and, during the following half-century, became some kind of benchmark problem even in the further accumulation of PIO databases. A useful while concise description may be found in Klyde et al. (1996). It appears that the basic block diagram of Fig 3 is applicable here and the flight data show that in that unpowered glide flown the pilot was compliant hence $H_c(s) \equiv K_p$. Also the flight data show that the actuator was operating in the highly saturated region; moreover all subsequent reference showed that “not only do all of the applied Category I criteria indicate that the X-15 would not be susceptible to PIO but also the aircraft was found to be level 1 for most of the applied handling quality measures”; moreover, the experimental data show that the instability frequency for the linear system with a synchronous pilot loop closure is 5.31 rad/sec, while the observed PIO frequency was 3.3 rad/sec.

A. We have thus the case of an obvious PIO II case. This case will be analyzed using the frequency domain inequalities discussed above. According to Amato et al. (2001) we shall have

$$H_c(s) = \frac{3.476(s + 0.0292)(s + 0.883)}{(s^2 + 0.019s + 0.01)(s^2 + 0.8418s + 5.29)} \tag{11}$$

The time constant of the actuator is 0.04 sec. Since the slope of the saturable actuator equals 1 for unsaturated case, the open loop transfer function of the linear (unsaturated) system of Fig. 3 is

$$H_b(s) = \frac{K}{(s + 25)}H_c(s) \tag{12}$$

where $K = 25 \cdot K_p$

A standard technique would be to compute root locus for the above open loop transfer function.

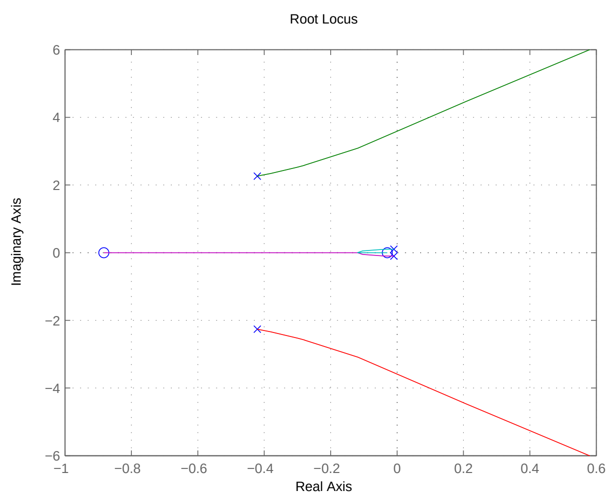

Fig. 4. Root locus for linear X15 PIO I model

We shall discuss in brief the root loci of fig.4. Clearly the far left real pole goes to $-\infty$ on the real axis; the starting dominant complex poles $-0.0095 \pm 0.1j$ will approach asymptotically the two real zeros -0.0292 and -0.883. The other starting complex poles $-0.4207 \pm 2.3\jmath$, while not being dominant, migrate to the RHP thus generating *oscillatory instability*; this instability corresponds to the imaginary poles $\pm 3.551\jmath$ and the pilot gain $K_p = 2.28$. It is higher than the value prescribed by Amato et al. (2001) which was 2.0425. On the other hand the frequency 3.551 is lower than 5.31 reported by Amato et al. (2001) but closer to the PIO II frequency 3.3 reported by Klyde et al. (1996). The only explanation at hand is that the cited references are using experimental data and there is some mismatch between real data and the reported model.

B. We shall turn now to the application of the frequency domain inequalities for the following transfer function of the linear part

$$H_L(s) = \frac{(s+0.0292)(s+0.883)}{s(s^2+0.019s+0.01)(s^2+0.8418s+5.29)} \tag{13}$$

Clearly we are here in the first critical case since we have a simple pole at $s = 0$.

The Popov frequency domain inequality for (13) means

$$\frac{1}{k} + \Re e\ (1 + \jmath\omega\theta) H_L(j\omega) \geq 0 \tag{14}$$

for some $\theta \geq 0$; note that in the usual graphical interpretation of (14) we have to introduce the Popov locus by

$$X_P(\omega) = \Re e\ H_L(\jmath\omega)\ ,\ Y_P(\omega) = \omega \Im m\ H_L(\jmath\omega) \tag{15}$$

thus obtaining

$$\frac{1}{k} + X_P(\omega) - \theta Y_P(\omega) \geq 0\ ,\ \forall \omega \in \mathbb{R} \tag{16}$$

The Yakubovich frequency domain inequality (6) will be considered for globally Lipschitz functions satisfying

$$0 \leq \frac{\varphi(\sigma_1) - \varphi(\sigma_2)}{\sigma_1 - \sigma_2} \leq K \tag{17}$$

with $K > 0$ as above. Since this means $\underline{\nu} = 0, \overline{\nu} = \overline{\varphi} = K$ the frequency domain inequality (6) becomes Răsvan et al. (2010)

$$\Re e\ (1 + \jmath\omega\theta + \gamma\omega^2)\left(\frac{1}{K} + H_L(\jmath\omega)\right) \geq 0 \tag{18}$$

This inequality also may be written as

$$\frac{1}{K} + X_Y(\omega) - \theta Y_Y(\omega) \geq 0 \tag{19}$$

where the modified transfer loci family (with respect to the parameter $\gamma > 0$) is defined by

$$X_Y(\omega) = \Re e\ H_L(\jmath\omega)\ ,\ Y_Y(\omega) = \frac{\omega}{1+\gamma\omega^2} \Im m\ H_L(\jmath\omega) \tag{20}$$

In order to judge sharpness of the frequency domain criteria for absolute stability, we turned to the *Aizerman problem* in the given case: to find the maximal sector in the linear case, the corresponding frequency of the oscillatory instability and, then, make a comparison to the nonlinear case. With respect to this, the root locus for $H_L(s)$ given by (13) and announcing, as mentioned, the first critical case, gave $K = 3.818$ and the oscillatory instability at $\omega = 2.13$.

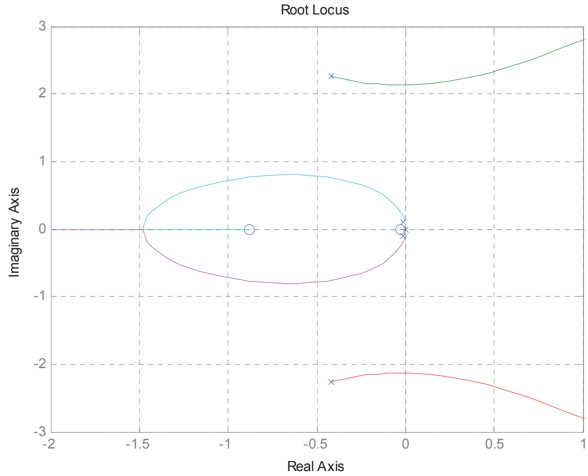

Fig. 5. Root locus for linearized X15 PIO II model

We next applied the frequency domain inequalities. According to the "parsimony principle" which here means that we start with a minimum number of free parameters in the inequality, there was considered first the Popov inequality; as it may be seen in fig. 6, the chosen scale required a zoom at relatively "high" frequencies and it clearly appears as obvious that the answer to the Aizerman problem would be negative i.e. we shall have $K < 3.818$; more precisely $K_{max} \approx 1/7.5$.

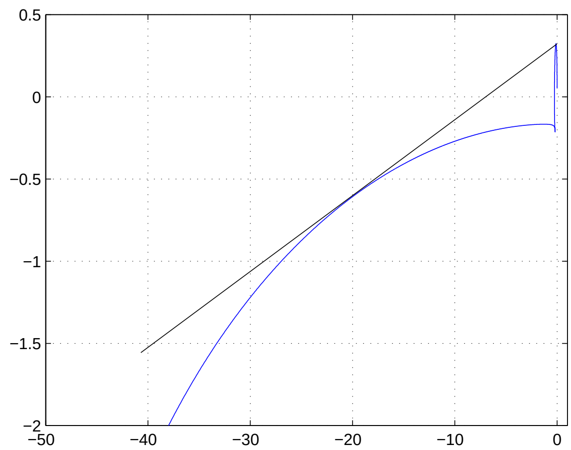

Fig. 6. Popov locus for X15 PIO II model

Therefore we turned to the Yakubovich criterion. The factor $(1 + \gamma\omega^2)^{-1}$ will attenuate the peak of fig.6 - diagram that corresponds to $\gamma = 0$: larger is $\gamma > 0$, lower is the peak - see fig.7,fig.8, corresponding to $\gamma = 0.1$ and $\gamma = 0.5$ respectively

Observe from fig.9 corresponding to $\gamma = 1$ that the peak almost disappeared and this shifted the Popov Yakubovich line crossing to the right of the real axis, thus increasing the admissible K_{max}; here $K_{max} \approx 3$ but graphics errors may be

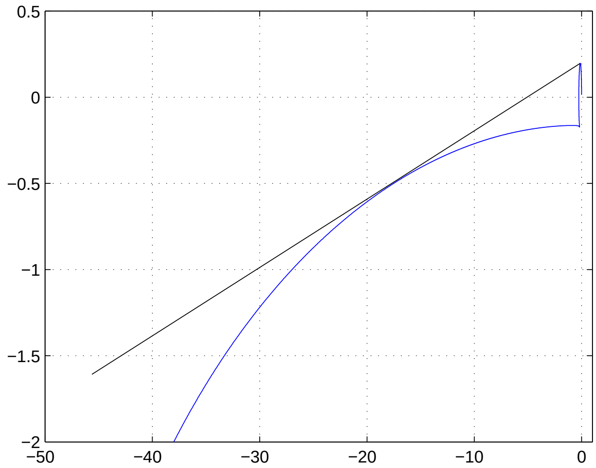

Fig. 7. Yakubovich locus for X15 PIO II model ($\gamma = 0.1$)

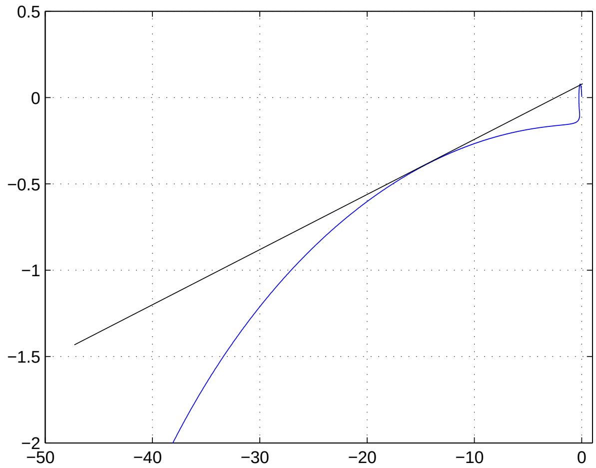

Fig. 8. Yakubovich locus for X15 PIO II model ($\gamma = 0.5$)

suspected. However, the comparison to $K_{max} \approx 1/7.5$ speaks for the improvement due to the application of the Yakubovich criterion.

4. CONCLUSIONS AND PERSPECTIVE

The research displayed in this paper represents the application to a real case analysis of the philosophy that considers PIO I and PIO II from the same point of view - that of the absolute stability and of the Aizerman problem. This underlying idea of the present research (and not of the only one) is, as mentioned in the introduction, to make use of the fact that in PIO II like in PIO I the airframe and pilot models are linear but the position and rate limiters are activated hence at least one saturation nonlinearity is involved. The saturation is a "weak" nonlinearity: sector restricted, monotone, non-decreasing, piecewise differentiable. Consequently, discussion of PIO proneness implies an absolute stability problem and unitary treatment of PIO I and PIO II sends to the Aizerman problem, where the linear stability (Hurwitz) and absolute stability sectors are put in comparison.

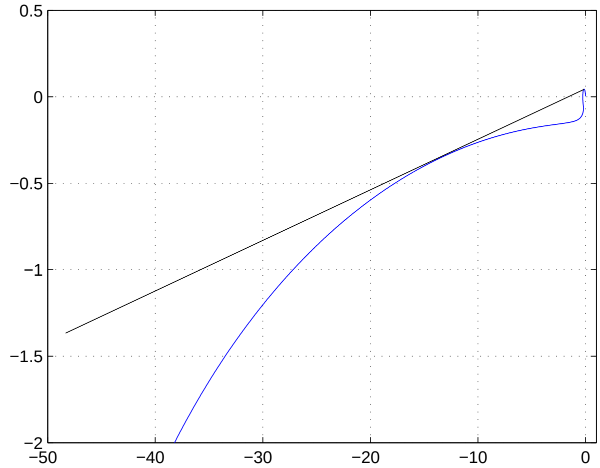

Fig. 9. Yakubovich locus for X15 PIO II model ($\gamma = 1$)

From the methodology point of view this requires implication of the absolute stability tools. Since all aircraft databases for PIO analysis use frequency domain characteristics, it is only natural to use frequency domain inequalities. A case study for X15 landing flare incident showed the role of the criteria with many parameters e.g. the criterion of Yakubovich for slope restricted nonlinearity to reduce the gap between PIO I and PIO II estimates. Worth mentioning that we use a version of the Yakubovich criterion extended to critical cases since $H_L(s)$ in (13) had a simple pole at $s = 0$.

The encouraging results suggest that the research should continue.

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