Slope Restrictions. Multipliers. Stability Inequalities ⋆

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Abstract: It is considered an overview of the frequency domain inequalities for the absolute stability of the systems with monotone and slope restricted nonlinearities. It appears that the same type of multiplier is associated with different augmentations of the state space and this fact explains various additional assumptions accompanying the stability inequalities. These inequalities are applied to the PIO II problem in aircraft dynamics where the feedback structure of the absolute stability contains the saturation nonlinearity which is both non-decreasing and slope restricted.

Keywords: Absolute stability, Multipliers, Slope restrictions, Frequency domain

1. THE STARTING POINTS OF THE PROBLEM

We shall start from the standard system of ordinary differential equations

$$\dot{x} = Ax - b\phi(c^*x) \tag{1}$$

where the state vector x has dimension n, $\phi : \mathbb{R} \mapsto \mathbb{R}$ is a scalar continuous function and the constant coefficients A, b, c have appropriate dimensions.

A. We assume that ϕ is a sector restricted nonlinear function i.e. that it is subject to

$$\underline{\phi} \leq \frac{\phi(\sigma)}{\sigma} \leq \bar{\phi} \ , \ \phi(0) = 0 \tag{2}$$

Obviously (2) defines an entire class of functions; since each function of this class defines a system (1) when considered in its equations, one may say that (1) - (2) define an entire class of nonlinear systems. Since $\phi(0) = 0$ these systems have $x \equiv 0$ (the equilibrium at the origin) as solution. Asymptotic stability of this equilibrium is a standard problem of the Liapunov theory. Less standard is the requirement that global asymptotic stability should hold for all nonlinear functions subject to (2). This is some kind of robustness of the stability and, following an almost 70 years tradition, Bulgakov (1942), *absolute stability*.

We mention here one of the most recent applications of absolute stability is the so-called PIO II problem in aircraft dynamics - the P(ilot) I(n-the-loop) O(scillations) of the second category, defined by the activation of the position and rate limiters; this means that in the feedback structure composed of the airframe and the pilot dynamics, a nonlinearity of the saturation type occurs (Fig. 1)

The saturation function is of sector restricted type. On the other hand system (1) may be viewed as describing a feedback structure composed of a linear and a nonlinear block as follows

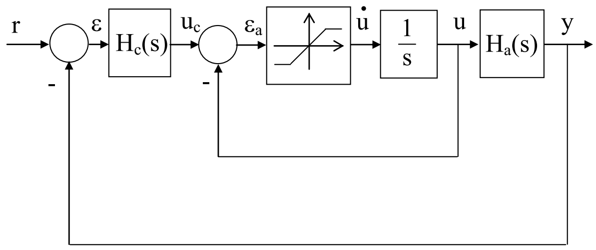

Fig. 1. System with rate limiter

$$\begin{aligned} &\dot{x} = Ax + b\mu_1(t) \ , \ \nu_1 = c^*x \ ; \\ &\nu_2 = \phi(\mu_2) \ ; \ \mu_2 = \nu_1 \ , \ \mu_1 = -\nu_2 \end{aligned} \tag{3}$$

(see also Fig.2)

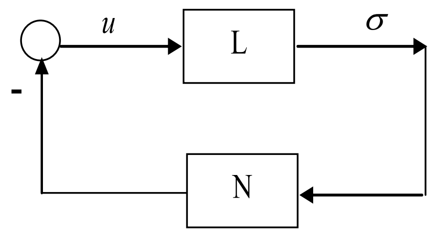

Fig. 2. Absolute stability feedback structure.

A straightforward approach to take in the PIO II problem is that of the absolute stability - Răsvan and Danciu (2010). But a problem occurs from the beginning- that of the sharpness of the results. Saturation is a specific sector restricted nonlinearity while the absolute stability techniques are valid for an entire class for nonlinearities, hence the stability conditions will be only sufficient i.e. lacking enough sharpness.

B. The sharpness problem of the absolute stability approaches has been considered from its early days. The so-called *Aizerman conjecture*, Aizerman (1949), says that the maximal absolute stability sector (2) coincides with the Hurwitz sector - the linear stability sector corresponding to $\phi(\sigma) = h\sigma$. This conjecture is valid for first order systems ($n = 1$), also for

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second order systems ($n=2$) except some limit situations as that described by a counter example due to Krasovskii (1952) but was clearly disproved for third order systems ($n=3$) by a celebrated counter example due to V. A. Pliss around 1957, Pliss (1958). Approximately around the same date Kalman (1957) started another conjecture: the Hurwitz stability sector coincides with the maximal sector of absolute stability for the so-called shape restricted nonlinear functions satisfying

$$\underline{\nu} \le \phi'(\sigma) \le \overline{\nu} \tag{4}$$

Slope restrictions clearly make the class of nonlinearities more narrow hence the result of Barabanov (1988) proving the validity of this conjecture for third order systems was somehow expected; at the same time i.e. in the same paper a procedure for constructing fourth order counterexamples to Kalman conjecture was proposed. Both conjectures disproved, there is still room for applications of the problem they generated: to test the sharpness of any method applied in absolute stability by comparison between the Hurwitz sector and the nonlinear stability sector provided by that method. The aim of this paper is to contribute to these methodological aspects in the case of the slope restrictions where several absolute stability criteria have been worked out. All of them have in common the so-called technique of the augmented state space Barabanov (2000). Our approach will be however an engineering one, based on frequency domain inequalities, frequency domain characteristics and frequency domain stability multipliers. The hyperstability theory, Popov (1973), is furnishing a philosophy as follows: to cope with "the usual point of view of the control engineer who likes to have at his disposal a wide range of elements capable of being combined in various ways to form control systems as complex as desired, but who does not like to burden his creative imagination with instability problems" Popov (1973), page 5.

2. ABOUT THE STABILITY MULTIPLIERS

We shall follow here the way of Krasovskii (1978): starting from the basic structure of Fig. 2 we perform equivalence transformations of it. Consider the equivalent structure of Fig. 3

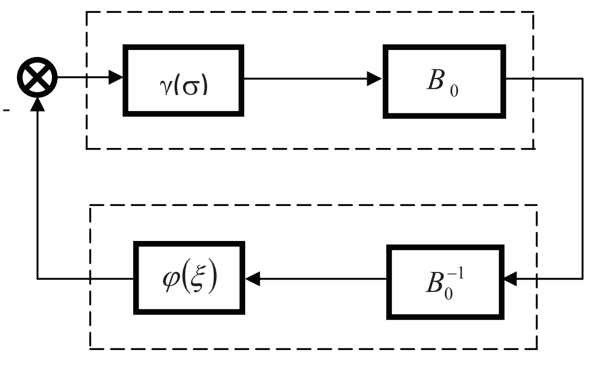

Fig. 3. Series augmented system.

If B_0 is a linear proper (causal) block, its inverse is also a proper i.e. causal block. This is the case, for instance, with the block describing the so-called Brockett Willems multiplier, whose transfer function is

$$\chi_{BW}(s) = \sum_1^p \delta_j \frac{s+\rho"_j}{s+\rho'_j+\rho"_j}\ ,\ \delta_j \ge 0\ ,\ \rho'_j > 0\ ,\ \rho"_j \ge 0 \tag{5}$$

But B_0 may be even improper: in fact the oldest stability multiplier - the Popov multiplier - is improper since it reads

$$\chi_P(s) = \alpha + \beta s \tag{6}$$

being thus a PD multiplier. Obviously B_0^{-1} is in this case a strictly proper linear block. While the theory of the absolute stability based on non-causal multipliers has attracted some researchers in the 70ies of the 20th century, only a few of the criteria obtained a broader use. The oldest of these criteria is that due to Yakubovich (1962): it corresponds to $\underline{\phi}=0$ and has the form

$$\begin{aligned} &\tau_1\left(\frac{1}{\phi} + \Re e\,\chi(\imath\omega)\right) + \tau_2 \Re e\ \imath\omega\chi(\imath\omega) + \\ &+\tau_3\omega^2 \Re e(1+\underline{\nu}\chi(-\imath\omega))(1+\overline{\nu}\chi(\imath\omega)) \ge 0 \end{aligned} \tag{7}$$

with $\chi(s) = c^*(sI-A)^{-1}b$ and for some real numbers τ_i, $\tau_1 \ge 0$, $\tau_3 \ge 0$.

If additionally we take $\underline{\nu}=0$ i.e. the admissible functions are also non-decreasing then (7) becomes:

$$\Re e\ (\tau_1 + \tau_2\imath\omega + \tau_3\omega^2)\chi(\imath\omega) + \frac{\tau_1}{\phi} + \tau_3\omega^2 > 0 \tag{8}$$

and one may recognize the multiplier

$$Z(s) = \tau_1 + \tau_2 s - \tau_3\overline{\nu}s^2 \tag{9}$$

The next criterion was concerned with slope restrictions only, see Barabanov and Yakubovich (1979), i.e. only the restrictions (4) are taken into account. The frequency domain criterion takes the form

$$\Re e\left\{(1+\underline{\nu}\chi(-\imath\omega))(1+\overline{\nu}\chi(\imath\omega)) - \frac{\tau}{\imath\omega}\chi(\imath\omega)\right\} \ge 0 \tag{10}$$

and if $\underline{\nu}=0$ then (10) becomes

$$\Re e\left(1+\overline{\nu}\chi(\imath\omega) - \frac{\tau}{\imath\omega}\chi(\imath\omega)\right) \ge 0 \tag{11}$$

which suggest the PI multiplier

$$Z(s) = \overline{\nu} - \frac{\tau}{s} = \overline{\nu}\left(1 - \frac{\tau}{\overline{\nu}}\frac{1}{s}\right) \tag{12}$$

This multiplier is causal. Several years later the same case was considered in Singh (1984)with a multi- variable counterpart in Haddad and Kapila (1995), Haddad (1997); the slope restrictions were $(0, \overline{\nu})$ and the frequency domain inequality was of the type (11) with τ changed in $-\tau$. Since the sign of τ is not specified we have obviously the same inequality.

Moreover, if we take $\tau_1 = 0$ in (8) and divide the inequality by $\omega^2 > 0$ we rediscover (11). It appears that from the point of view of the frequency domain inequality all criteria are identical. There exist however several differences connected with proof techniques and they introduce additional assumptions which have corresponding effects on the practical stability conditions.

3. THE AUGMENTED SYSTEMS

We mention from the beginning that a good overall reference is Barabanov (2000). Here we focus on the systems associated to the cases described in the previous section. In the case of Yakubovich (1965a,b) (in fact this was mentioned even in Yakubovich (1962)) the augmented system was defined by the state variables

$$z = x \ , \ \zeta = -\phi(c^* x) \tag{13}$$

what sends to the $(n+1)$-dimensional system

$$\begin{aligned} \dot{z} &= Az + b\zeta \\ \dot{\zeta} &= -\phi'(c^* z)c^*(Az + b\zeta) \end{aligned} \tag{14}$$

Obviously this system has the prime integral

$$\zeta(t) + \phi(c^* z(t)) \equiv \text{const} \tag{15}$$

hence its dimension may be reduced by one; moreover if the solutions of (14) are viewed on the invariant set $\zeta + \phi(c^* z) \equiv 0$ - suggested by (13) - then $z(t) \equiv x(t)$ provided $z(0) = x(0)$. This extended system was considered in Barbălat and Halanay (1974) for the case of several nonlinear functions. Consider now the approach of Barabanov and Yakubovich (1979); here the new state variables are

$$z = Ax - b\phi(c^* x) \ , \ \zeta = -\phi(c^* x) \tag{16}$$

and unlike (13) here $z = \dot{x}$. From here the following is obtained

$$\begin{aligned} \dot{z} &= Az + b\mu(t) \\ \dot{\zeta} &= \mu(t) \ , \ \mu(t) = -\phi'(c^* x(t))c^* z \end{aligned} \tag{17}$$

If $\det A \neq 0$ then we may compute $c^* x = c^* A^{-1}(z - b\zeta)$ to obtain the$(n+1)$-dimensional system

$$\begin{aligned} \dot{z} &= Az + b\zeta \\ \dot{\zeta} &= -\phi'(c^* A^{-1}(z - b\zeta))c^* z \end{aligned} \tag{18}$$

with the prime integral

$$\zeta(t) + \phi(c^* A^{-1}(z(t) - b\zeta(t))) \equiv \text{const} \tag{19}$$

The third approach of Singh (1984), Haddad and Kapila (1995) is based on "differentiating the initial system"; this means

$$z = Ax - b\phi(c^* x) \ , \ \zeta = c^* x \tag{20}$$

hence

$$\begin{aligned} \dot{z} &= Az - b\phi'(\zeta)c^* z \\ \dot{\zeta} &= c^* z \end{aligned} \tag{21}$$

with the prime integral

$$\zeta(t) - c^* A^{-1} z(t) - c^* A^{-1} b\phi(\zeta(t)) \equiv \text{const} \tag{22}$$

Not only that (15) are the simplest in defining the new state variables but also the "return" to the basic system ((13) *via* the associated prime integral generating a family of invariant sets is much simpler. This suggests, especially when thinking to the assumption $\det A \neq 0$ that slope restrictions are taken into account in a more natural way if considered together with the sector restrictions.

4. SOME APPLICATIONS

Several applications with purely mathematical character may be found in Barbălat and Halanay (1970, 1971, 1974), Răsvan (2007).

A. We consider first the previously mentioned celebrated counterexample of Pliss. In this case the transfer function of the linear part is

$$\chi(s) = \frac{1}{s+1+a} + \frac{s-1}{s^2+1} \ , a > 0 \tag{23}$$

which is irreducible and has two poles on $\imath\mathbb{R}$. It is quite easily checked that the Hurwitz sector is given by $0 < h < (1+a)/a$. Consequently the maximal achievable result for the absolute stability sector is subject to $\underline{\phi} \geq \underline{\nu} > 0$ and $\overline{\phi} \leq \overline{\nu} < (1+a)/a$.

Here as elsewhere we shall follow the philosophy of some *parsimony principle*: to use as few free parameters as necessary i. e. the stability multiplier should be as simple as possible. We are guided by our experience which "tells" that more free parameters are in use, more difficult is to manipulated them in a reasonable way.

Application of the Popov criterion requires to take in (7) $\tau_3 = 0$ to find

$$\frac{1}{\overline{\phi}} + \Re e(1 + \imath\omega\theta)\chi(\imath\omega) = \frac{1}{\overline{\phi}} + \frac{1+a+\omega^2\theta}{(1+a)^2+\omega^2} + \frac{1+\omega^2\theta}{\omega^2-1} \geq 0 \tag{24}$$

The only choice for θ is $\theta = -1$ to find

$$\overline{\phi} < 1/2 < (1+a)/a$$

Next, the case when only slope restrictions are taken into account does not apply for the Yakubovich criterion since it would require in (7) both $\tau_1 = \tau_2 = 0$, the inequality thus lacking any free parameter. Consequently we shall consider both sector and slope restrictions:

$$\underline{\phi} = \overline{\nu} = 0 \ , \ \overline{\nu} = \infty \ , \ \theta = \tau_2/\tau_1 \ , \ \beta = \tau_3/\tau_1$$

Then (7) reads

$$\frac{1}{\overline{\phi}} + \Re e(1 + \imath\omega\theta + \beta\omega^2)\chi(\imath\omega) \geq 0 \tag{25}$$

that is

$$\frac{1}{\overline{\phi}} + \frac{(1+a)(1+\omega^2\beta)+\theta\omega^2}{(1+a)^2+\omega^2} - \frac{1+\omega^2(\beta+\theta)}{1-\omega^2} \geq 0$$

and a necessary choice is $\beta + \theta = -1$ hence $\theta = -1 - \beta < 0$. Further an elementary computation shows that by choosing $\beta > 2/a$ the inequality (25) holds provided $1/\overline{\phi} - a/(1+$

$a) > 0$ what recovers the Hurwitz sector for all non-decreasing nonlinear functions of this sector.

B. The next application accounts for a preliminary computation for PIO II prevention in the short period longitudinal motion of the so-called ADMIRE standard model. With the specific notations we have

$$\begin{aligned} &\ddot{\alpha} - M_q\dot{\alpha} - M_\alpha \alpha - M_\delta \delta_c = 0 \\ &\dot{\delta}_c = \omega_e \psi(-k_\alpha \alpha - k_q \dot{\alpha} - \delta_c) \end{aligned} \tag{26}$$

with α - the incidence angle and δ_c - the control deflection angle. The function $\psi(\varepsilon)$ is the saturation function

$$\psi(\varepsilon) = \begin{cases} V_L \quad , \quad |\varepsilon| > \varepsilon_L \\ \dfrac{V_L}{\varepsilon_L}\varepsilon \quad , \quad |\varepsilon| \le \varepsilon_L \end{cases} \tag{27}$$

In order to estimate the Hurwitz sector we take $\psi(\sigma) = \gamma\sigma$ with $\gamma > 0$. The characteristic equation of the linear system thus obtained is

$$\begin{aligned} &D(\lambda) \equiv \lambda^3 + (\omega_e\gamma - M_q)\lambda^2 + (\omega_e\gamma(M_\delta k_q - M_q) - M_\alpha)\lambda + \\ &+\omega_e\gamma(M_\delta k_\alpha - M_\alpha) = 0 \end{aligned} \tag{28}$$

Since $k_\alpha \ge 0, k_q \ge 0, M_q < 0$ we shall have $A_q = M_\delta k_q - M_q > 0$ from the first Stodola inequality. The Hurwitz sector will be defined by $\omega_e\gamma > \xi_+$, where ξ_+ is the positive root of the trinomial

$$A_q\xi^2 - (M_\alpha + A_q M_q + A_\alpha)\xi + M_\alpha M_q = 0 \tag{29}$$

We “rotate” the sector by introducing

$$\phi(\sigma) = -\gamma_+\sigma - \psi(-\sigma) \tag{30}$$

where $\omega_e\gamma_+ = \xi_+$. We obtain in this way a feedback structure as in Fig.2 with the nonlinear function subject to

$$-\gamma_+ < \frac{\phi(\sigma)}{\sigma} \le -\gamma_+ + \frac{V_L}{\varepsilon_L} \ , \ -\gamma_+ < \phi'(\sigma) \le -\gamma_+ + \frac{V_L}{\varepsilon_L} \tag{31}$$

and the transfer function of the linear part of the form

$$\chi(s) = \omega_e \frac{s^2 + A_q s + A_\alpha}{(s + \xi_+ - M_q)(s^2 + A_q\xi_+ - M_\alpha)} \tag{32}$$

This transfer function has minimal phase, a real negative pole and a pair of poles on $\imath\mathbb{R}$. Considering the Popov frequency domain inequality

$$\Re e \ (1 + \imath\omega\theta)\chi(\imath\omega\theta) \ge 0 \tag{33}$$

accounting for a possible infinite absolute stability sector (that recovers the entire Hurwitz sector), the pair of imaginary poles gives the unique choice of

$$\bar{\theta} = \frac{A_\alpha p_1 + (A_q - p_1)\omega_0^2}{\omega_0^2(A_q p_1 + \omega_0^2 - A_\alpha)} > 0 \tag{34}$$

where we denoted $p_1 = \xi_+ - M_q > 0$, $\omega_0^2 = A_q\xi_+ - M_\alpha > 0$. This choice and the fact that $\bar{\theta} > 0$ gives

$$\Re e \ (1 + \imath\omega\theta)\chi(\imath\omega\theta) = \frac{\bar{\theta}\omega^2 + A_\alpha p_1 \omega_0^{-2}}{\omega^2 + p_1^2} > 0 \ , \ \forall\omega \tag{35}$$

Since the entire Hurwitz sector has been recovered, we deduce that (26) is absolutely stable in the sector (γ_+, ∞). But this sector is “violated” by the specific nonlinear function (27). We are thus stressed to find an invariant set of the state space where this sector is not violated. From the graphical condition $|\varepsilon| < V_L/\gamma_+$ the following condition is obtained

$$(k_\alpha\alpha + k_q\dot{\alpha} + \delta_c)^2 < (V_L/\gamma_+)^2 \tag{36}$$

and we need the largest invariant set included in (36). The most “at hand” invariant sets have the form $V(x) < c$ where x is the state vector, $V : \mathbb{R} \mapsto \mathbb{R}_+$ a suitable Liapunov function and $c > 0$ the largest possible such that

$$\begin{aligned} &\left\{ \sup_{c>0}\{x \in \mathbb{R}^3 : V(x) < c\} \right\} \subset \\ &\subset \{x \in \mathbb{R}^3 : (k_\alpha\alpha + k_q\dot{\alpha} + \delta_c)^2 < (V_L/\gamma_+)^2\} \end{aligned} \tag{37}$$

Paradoxically, we need here a Liapunov function while in aircraft dynamics and PIO analysis all available data are expressed in the frequency domain. Fortunately we may use the Yakubovich Kalman Popov lemma to associate to the frequency domain inequality (33) - and (35) - a Liapunov function of the form

$$V(x) = x^* H x + \bar{\theta} \int_0^{c^* x} \phi(\lambda) \mathrm{d}\lambda \tag{38}$$

where $\bar{\theta} > 0$ is that of (34) while H is a result of solving some Linear Matrix Inequalities.

C. Another application is the analysis of the PIO II proneness for the roll attitude of the lateral directional motion of a generic aircraft, see Klyde et al. (1995). The mathematical model reads

$$\begin{aligned} &\ddot{\phi} + \frac{1}{T_R}\dot{\phi} = L_a\delta_a \\ &\dot{\delta}_a = \omega_e\psi(-k_\phi\phi - k_p\dot{\phi} - \delta_a) \end{aligned} \tag{39}$$

where ϕ is the bank angle and δ_a the aileron deflection angle; $\psi(\cdot)$ is again given by (27). The transfer function of the linear part is

$$\chi(s) = \omega_e\left(\frac{1}{s} + L_a\frac{k_\phi + k_p s}{s^2(s + 1/T_R)}\right) \tag{40}$$

hence it is in the critical case of the double pole. The Popov criterion holds for the “infinite” parameter i.e. for

$$\omega\Im m \ \chi(\imath\omega) > 0 \ \Leftrightarrow \ \frac{T_R^2\omega^2 + 1 + L_a T_R(k_p - k_\phi T_R)}{T_R^2\omega^2 + 1} > 0 \tag{41}$$

which holds for

$$1 + L_a T_R (k_p - k_\phi T_R) > 0 \tag{42}$$

5. CONCLUSIONS AND PERSPECTIVES

Throughout this paper there were presented two kinds of problems - theoretical and applied. The theoretical issues account for the specific features of the stability multipliers which are connected with monotonicity and slope restrictions of the nonlinear functions. The fact the same structure of the frequency domain inequality is obtained under various proof assumptions is a sound explanation for the additional assumptions accompanying the frequency domain inequalities.

At the same time this analysis shows the role of the non-causal stability multipliers which are still not enough investigated. On the other hand it is an interesting coincidence that such an important application as PIO II may be embedded within the absolute stability problem and that the saturation function is both non-decreasing and slope-restricted. Moreover, since the aircraft dynamics databases are expressed in frequency domain, this is a rather important field of applications of the frequency domain inequalities. Once more the almost 70 year old field of the absolute stability is rewarding for both theoretical and applied research.

REFERENCES

M. A. Aizerman. On a problem concerning stability ”in the large” of dynamical systems (russian). *Usp.Mat.Nauk*, 4(4): 187–188, August 1949.

N. E. Barabanov. About the problem of kalman (russian). *Sib. Mat. Ž*, 29(3):3–11, June 1988.

N. E. Barabanov. The state space extension method in the theory of absolute stability. *IEEE Trans. Aut. Contr.*, 45(12): 2335–2339, December 2000.

N. E. Barabanov and V. A. Yakubovich. Absolute stability of control systems having one hysteresis-like nonlinearity(in russian). *Avtomat. i Telemekhanika*, 40:5–12, December 1979.

I. Barbălat and A. Halanay. Applications of the frequency-method to forced nonlinear oscillations. *Mathem. Nachr.*, 44: 165–179, 1970.

I. Barbălat and A. Halanay. Nouvelles applications de la méthode fréquentielle dans la théorie des oscillations. *Rev. Roum. Sci. Techn. Electrotechn. et Energ.*, 16:689–702, October-December 1971.

I. Barbălat and A. Halanay. Conditions de comportement “presque linéaire” dans la théorie des oscillations. *Rev. Roum. Sci. Techn. Electrotechn. et Energ.*, 19:961–979, April-June 1974.

B. V. Bulgakov. Self-sustained oscillations of control systems (russian). *DAN SSSR*, 37(9):283–287, 1942.

W. M. Haddad. Correction to “absolute stability criteria for multiple slope-restricted monotonic nonlinearities”. *IEEE Trans. on Autom. Control*, 42(4):591, April 1997.

W. M. Haddad and V. Kapila. Absolute stability criteria for multiple slope-restricted monotonic nonlinearities. *IEEE Trans. on Autom. Control*, 40(2):361–365, February 1995.

R. E. Kalman. Physical and mathematical mechanisms of instability in nonlinear automatic control systems. *Trans. ASME*, 79(3):553–563, April 1957.

D. H. Klyde, B. L. Aponso, D. G. Mitchell, and R. H. Hoh. Development of roll attitude quickness criteria for fighter aircraft. Paper 95-3205-CP, AIAA, 1995.

N. N. Krasovskii. Theorems concerning stability of motions determined by a system of two equations (russian). *Prikl. Mat.Mekh.(PMM)*, 16(5):547–554, October 1952.

N. N. Krasovskii. Some system theory ideas connected with the stability problem. *Cybernetics and Systems (form. Journ. Cybernetics)*, 8(2):203–215, April 1978.

V. A. Pliss. *Some problems of the theory of stability of motion in the large (Russian)*. Leningrad State Univ.Publ.House, Leningrad, USSR, 1958.

V. M. Popov. *Hyperstability of Control Systems*. Springer Verlag, Berlin-Heidelberg-New York, 1st edition, 1973.

Vl. Răsvan. A new dissipativity criterion - towards yakubovich oscillations. *Int. J. Rob. Nonlin. Contr.*, 17:483–495, 2007.

Vl. Răsvan and Daniela Danciu. Pio ii - a unifying point of view. In *Proc. Int. Joint Confer. Comput. Cyb. Techn. Inform. ICCC-CONTI 2010*, pages 17–21, Timişoara, Romania, May 2010.

V. Singh. A stability inequality for nonlinear feedback systems with slope-restricted nonlinearity. *IEEE Trans. on Autom. Control*, 29(8):743–744, August 1984.

V. A. Yakubovich. Frequency domain conditions for absolute stability of nonlinear control systems (russian). In *Proc. Inter-Univ. Confer. on Appl. Stab. Theory and Anal. Mech.*, pages 135–142, Kazan, USSR, 1962.

V. A. Yakubovich. Frequency domain conditions of absolute stability and dissipativeness of control systems with a single differentiable element(in russian). *Dokl. Akad. Nauk SSSR*, 160(2):298–301, April 1965a.

V. A. Yakubovich. Matrix inequalities method in the theory of stability of controlled systems ii. absolute stability in a class of nonlinearities with the restrictions on the derivative(in russian). *Avtomat. i Telemekhanika*, 29:577–583, April 1965b.