An evergreen benchmark in absolute stability - the gyroscopic pendulum I: The Aizerman problem

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Abstract: It has been established by now that the problem of the absolute stability originates from a paper of B.V. Bulgakov published in 1942. The engineering application generating the problem was the so called *gyroscopic pendulum*, with various applications in stabilization of ships, aircrafts or armored cars. Even if nowadays the technical implementation of this device might be completely different, the dynamics is essentially unchanged. For this reason, it became in the next decades a genuine benchmark in the absolute stability. In this paper there are considered stability and oscillations of the gyroscopic pendulum based on both achievements of the absolute stability - the Liapunov function and the frequency domain inequalities.

Keywords: Absolute stability, Frequency domain inequalities, Liapunov function

1. INTRODUCTION AND PROBLEM STATEMENT.

A. In a rather interesting advanced textbook Voronov (1979) it is mentioned that the problem of the determination of the stability of a system containing a linear dynamical block and a static nonlinear one (e.g. a sector restricted non-linearity) in feedback connection was formulated for the first time by B.V. Bulgakov in two papers, the earliest being from 1942 (Bulgakov (1942, 1946)). The problem of the absolute stability is thus almost 70 years old.

The idea of considering sector restricted nonlinearities without any other specific additional information appears for the first time in these papers. The approach of Bulgakov (1942, 1946) was to consider that instability occurs *via* some self oscillating regime and, consequently, the author sought conditions for the absence of self sustained oscillations using the approach of the small parameter. With respect to this we have to mention here another paper of Bulgakov (1943).

In the research that followed only the way of specifying the nonlinearity as sector restricted was considered: the approach was that of a special Liapunov function, the pioneering paper being Lurie and Postnikov (1944). The overwhelming success (due to further development) of this method had as consequence the consideration of Lurie and Postnikov (1944) as the starting point of the problem while it had been in fact only the starting point of a more successful approach. But the papers of Bulgakov, while underestimated from this point of view (of the approach) are nevertheless important when seeking the necessary and sufficient conditions for absolute stability: the early approach of Pliss (1958) for the proof or the disproof of the Aizerman conjecture is an argument for this assertion.

B. We shall give below, following Bulgakov (1954), the basic equations of the gyroscopic pendulum with horizontal equilibrium, based on the momentum balance and assuming sufficiently small angular displacements

$$
\begin{aligned}
A\ddot{\alpha} + L\alpha + H\dot{\beta} &= 0 \\
B\ddot{\beta} + M\beta - H\dot{\alpha} &= 0
\end{aligned}
\tag{1}
$$

We have here a system with two angular degrees of freedom:

- α - the angular displacement of the pendulum with respect to the vertical line;
- β - the angular displacement of the gyroscope case with respect to its average position on its spindles that have the direction of the pendulum rod.

The constant parameters of the differential equations are as follows

- A - the inertia moment of the entire system with respect to the swing axis of the pendulum;
- B - the inertia moment of the gyroscope with his case with respect to the spindle axis;
- L - the coefficient of $L\alpha$ - the moment of the gravitation force of the entire system;
- M - the coefficient of the reacting moment $M\beta$ of the spring which maintains the gyroscope case in its average (equilibrium) position with respect to its spindle axis;
- H - the coefficient of the gyroscopic moments $H\dot{\alpha}$ and $-H\dot{\beta}$.

We introduce, following Bulgakov (1954) the rated parameters

$$
\rho = \sqrt{\frac{L}{A}}\ ,\ \sigma = \sqrt{\frac{M}{B}}\ ,\ \kappa = \frac{H}{A}\ ,\ \lambda = \frac{H}{B}
\tag{2}
$$

Next, the equations are completed with the terms $-\varepsilon\dot{\alpha}$ and $-\varepsilon\dot{\beta}$ accounting for the viscous friction, also with $-\psi_1(\dot{\alpha})$ and $-\psi_2(\dot{\beta})$ - the nonlinear friction forces at the bearings. As a result we obtain the following equations

$$
\begin{aligned}
\ddot{\alpha} + \varepsilon_1\dot{\alpha} + \rho^2\alpha + \kappa\dot{\beta} &= -\psi_1(\dot{\alpha}) \\
\ddot{\beta} + \varepsilon_2\dot{\beta} + \sigma^2\beta - \lambda\dot{\alpha} &= -\psi_2(\dot{\beta})
\end{aligned}
\tag{3}
$$

We have here a nonlinear system with two nonlinear elements which will be studied from the mathematical point of view. It is important to mention here that all parameters were supposed positive but we shall, however, accept negative values for κ and λ; this has the significance of the sign inversions for the gyroscopic feedback and such possibility exists in the modern gyroscopic systems which may be electronic or even software implemented.

C. We shall state here the problems we intend to analyze, completing the early development of Bulgakov (1954). The nonlinear functions $\psi_i : \mathbb{R} \mapsto \mathbb{R}$ are supposed to be sector restricted

$$\underline{\delta}_i \le \frac{\psi_i(\nu)}{\nu} \le \overline{\delta}_i \ , \ \psi_i(0) = 0 \ \ i = 1, 2 \tag{4}$$

The absolute stability problem reads as follows

Problem 1. (Absolute stability) Given the system (3) where the continuous functions $\psi_i : \mathbb{R} \mapsto \mathbb{R}$ are subject to (4), find conditions on (ε_i, ρ^2, σ^2, κ, λ, $\underline{\delta}_i$, $\overline{\delta}_i$) in order that the equilibrium at the origin $\alpha = \dot{\alpha} = \beta = \dot{\beta} = 0$ should be globally asymptotically stable for all functions ψ_i satisfying (4).

A comment appears as necessary. In Bulgakov (1942, 1943, 1946) there were considered several applications with nonlinear functions confined to a sector (4) Two of them, having very similar equations, got the names "first" and "second" problems of Bulgakov (Letov (1961)). None of them coincides with with the absolute stability problem for (3) since in those cases there is a single nonlinear function.

The next problem we shall involve in our analysis will be the Aizerman problem. This problem arises from the so called *Aizerman conjecture* that had been stated almost simultaneously with the problem that generated it - the absolute stability (Aizerman (1946, 1949)). It had been clear from the very beginning that absolute stability meant global asymptotic stability for all functions of the sector - linear and nonlinear. Were only linear functions considered, the maximal stability sector would be given by some linear stability criterion e.g. the Routh-Hurwitz criterion. The *conjecture of Aizerman* reads as follows: *the Hurwitz and absolute stability sectors coincide*. This conjecture was disproved by Pliss (1958). The modified Aizerman conjecture - the conjecture of Kalman which considered only slope restricted nonlinearities - being also disproved by Barabanov (1988), it was replaced by the problem - comparison of the Hurwitz sector, considered as maximal, with the absolute stability sector - in order to have an estimate of the "degree of conservatism" i.e. of the gap between the sufficient conditions and the necessary and sufficient conditions of stability, the "sharpness".

Our development will be concerned with the Aizerman problem for the gyroscopic pendulum and, therefore, is organized as follows. First the linearized system is considered and the Hurwitz sectors are determined. Next, a quadratic, energy like Liapunov function is associated and some absolute stability conditions are obtained. Since the most general (i.e. giving closest to the necessary and sufficient stability conditions) Liapunov function of the form "quadratic form plus integral of the nonlinear functions" is prescribed by the Popov frequency domain inequality *via* the positiveness theory (Yakubovich Kalman Popov lemma), it is considered the Popov inequality for the case of two nonlinear functions. Using the linear matrix equalities of the lemma, the required Liapunov function is constructed. This most general function *coincides with the energy like one* which thus turns to be the most general in this case. The problem of Aizerman is solved in an "almost positive" way i.e. the linear and nonlinear stability conditions coincide provided some robustness assumption on the linear subsystem is made.

2. THE STABILITY CONDITIONS FOR THE LINEARIZED SYSTEM.

We shall start from the equations (3) and since we are interested in the maximal stability sectors we introduce the new nonlinear functions

$$\varphi_i(\nu) = \varepsilon_i \nu + \psi_i(\nu) \ , \ i = 1, 2 \tag{5}$$

and deal with the modified but equivalent system

$$\begin{aligned} &\ddot{\alpha} + \rho^2 \alpha + \kappa \dot{\beta} = -\varphi_1(\dot{\alpha}) \\ &\ddot{\beta} + \sigma^2 \beta - \lambda \dot{\alpha} = -\varphi_2(\dot{\beta}) \end{aligned} \tag{6}$$

Consider first $\varphi_1(\nu) = \varphi_2(\nu) \equiv 0$. The remaining linear system has the following characteristic polynomial

$$\begin{vmatrix} s^2 + \rho^2 & \kappa s \\ -\lambda s & s^2 + \sigma^2 \end{vmatrix} = (s^2 + \rho^2)(s^2 + \sigma^2) + \lambda \kappa s^2 \tag{7}$$

hence if s_k is a root then $-s_k$ is also a root (characteristic number of the system). Considering the auxiliary equation

$$x^2 + (\rho^2 + \sigma^2 + \lambda\kappa)x + \rho^2\sigma^2 = 0, \tag{8}$$

it has two real negative roots provided

$$\lambda\kappa > -(\rho - \sigma)^2 \tag{9}$$

If (9) holds then (7) has two pairs of purely imaginary roots. This corresponds to the physical reality since (6) describes a mechanical system with nonlinear damping. Worth mentioning that the only important sign is of the product $\lambda\kappa$ for which (9) indicates that non-positive values are allowed.

Let now $\varphi_i(\nu) = \gamma_i \nu$, $\gamma_i > 0$: there is introduced some linear damping in the system, which takes the form

$$\begin{aligned} &\ddot{\alpha} + \gamma_1 \dot{\alpha} + \rho^2 \alpha + \kappa \dot{\beta} = 0 \\ &\ddot{\beta} + \gamma_1 \dot{\beta} + \sigma^2 \beta - \lambda \dot{\alpha} = 0 \end{aligned} \tag{10}$$

Its characteristic polynomial is

$$\begin{aligned} &\begin{vmatrix} s^2 + \gamma_1 s + \rho^2 & \kappa s \\ -\lambda s & s^2 + \gamma_2 s + \sigma^2 \end{vmatrix} = \\ &= (s^2 + \gamma_1 s + \rho^2)(s^2 + \gamma_2 s + \sigma^2) + \lambda\kappa s^2 = \\ &= s^4 + (\gamma_1 + \gamma_2)s^3 + (\gamma_1\gamma_2 + \rho^2 + \sigma^2 + \lambda\kappa)s^2 + \\ &+ (\gamma_1\sigma^2 + \gamma_2\rho^2)s + \rho^2\sigma^2 \end{aligned} \tag{11}$$

If (9) holds, all coefficients of the characteristic polynomials are strictly positive for $\gamma_i > 0$. The two determinant conditions of the Routh Hurwitz criterion are as follows

$$\begin{aligned} &(\gamma_1\gamma_2 + \rho^2 + \sigma^2 + \lambda\kappa)(\gamma_1 + \gamma_2) - (\gamma_1\sigma^2 + \gamma_2\rho^2) = \\ &= (\gamma_1 + \gamma_2)(\gamma_1\gamma_2 + \lambda\kappa) + \gamma_1\rho^2 + \gamma_2\sigma^2 > 0 \end{aligned}$$

which gives

$$\lambda\kappa > -\gamma_1\gamma_2 - \frac{\gamma_1\rho^2 + \gamma_2\sigma^2}{\gamma_1 + \gamma_2} \tag{12}$$

and

$$(\gamma_1\sigma^2 + \gamma_2\rho^2)[(\gamma_1 + \gamma_2)(\gamma_1\gamma_2 + \lambda\kappa) + \gamma_1\rho^2 + \gamma_2\sigma^2] -$$
$$-(\gamma_1 + \gamma_2)^2\rho^2\sigma^2 = \gamma_1\gamma_2(\rho^2 - \sigma^2)^2 +$$
$$+(\gamma_1 + \gamma_2)(\gamma_1\gamma_2 + \lambda\kappa)(\gamma_1\sigma^2 + \gamma_2\rho^2) > 0$$

which gives

$$\lambda\kappa > -\gamma_1\gamma_2\left[1 + \frac{(\rho^2 - \sigma^2)^2}{(\gamma_1 + \gamma_2)(\gamma_1\sigma^2 + \gamma_2\rho^2)}\right] \tag{13}$$

We have further

$$\frac{\gamma_1\rho^2 + \gamma_2\sigma^2}{\gamma_1\gamma_2} - \frac{(\rho^2 - \sigma^2)^2}{(\gamma_1\sigma^2 + \gamma_2\rho^2)} = \frac{(\gamma_1 + \gamma_2)^2\rho^2\sigma^2}{\gamma_1\gamma_2(\gamma_1\sigma^2 + \gamma_2\rho^2)} > 0$$

and this shows that (13) is the unique Routh Hurwitz condition, allowing negative values for the product $\lambda\kappa$: not only that negative values for λ, κ are allowed but these numbers are even allowed for opposite signs.

3. A NATURAL LIAPUNOV FUNCTION

Equations (6) show a system of two coupled mechanical oscillators. For each of them we can construct a Liapunov function as follows. Consider the coupled oscillator of e.g. the degree of freedom represented by α and write down its equations along some trajectory

$$\ddot{\alpha}(\tau) + \rho^2\alpha(\tau) \equiv -\varphi_1(\dot{\alpha}(\tau)), \tag{14}$$

multiply by $\dot{\alpha}(\tau)$ and integrate from 0 to t; after an integration by parts we deduce

$$\frac{1}{2}\left(\dot{\alpha}^2(t) + \rho^2\alpha^2(t)\right) \equiv \frac{1}{2}\left(\dot{\alpha}^2(0) + \rho^2\alpha^2(0)\right) - \\ - \int_0^t \varphi_1(\dot{\alpha}(\tau))\dot{\alpha}(\tau)\mathrm{d}\tau \tag{15}$$

which suggests the following energy function

$$V_1(\alpha, \dot{\alpha}) = \frac{1}{2}(\dot{\alpha}^2 + \rho^2\alpha^2) \tag{16}$$

For the other oscillator we shall have

$$V_2(\beta, \dot{\beta}) = \frac{1}{2}(\dot{\beta}^2 + \rho^2\beta^2) \tag{17}$$

Both functions are positive definite. For the interconnected system we shall define the composite Liapunov function as follows

$$V(\alpha, \dot{\alpha}, \beta, \dot{\beta}) = \tau_1 V_1(\alpha, \dot{\alpha}) + \tau_2 V_2(\beta, \dot{\beta}) > 0 \tag{18}$$

provided $\tau_i > 0$, $i = 1, 2$. Obviously $V(\alpha, \dot{\alpha}, \beta, \dot{\beta})$ is a quadratic diagonal form. Its derivative along the solutions of (6) is as follows

$$\frac{\mathrm{d}V^\star}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}V(\alpha(t), \dot{\alpha}(t), \beta(t), \dot{\beta}(t)) =$$
$$= \tau_1(\rho^2\alpha\dot{\alpha} + \dot{\alpha}\ddot{\alpha}) + \tau_2(\sigma^2\beta\dot{\beta} + \dot{\beta}\ddot{\beta}) =$$
$$= \tau_1\rho^2\alpha\dot{\alpha} + \tau_1\dot{\alpha}(-\rho^2\alpha - \kappa\dot{\beta} - \varphi_1(\dot{\alpha})) +$$
$$+ \tau_2\sigma^2\beta\dot{\beta} + \tau_2\dot{\beta}(-\sigma^2\beta + \lambda\dot{\alpha} - \varphi_2(\dot{\beta})) =$$
$$= -(\tau_1\kappa - \tau_2\lambda)\dot{\alpha}\dot{\beta} - \tau_1\varphi_1(\dot{\alpha})\dot{\alpha} - \tau_2\varphi_2(\dot{\beta})\dot{\beta}$$

Since $\varphi_i(\nu)\nu > 0$ the derivative is negative semi-definite provided the matrix

$$\begin{pmatrix} \tau_1 & \frac{1}{2}(\tau_1\kappa - \tau_2\lambda) \\ \frac{1}{2}(\tau_1\kappa - \tau_2\lambda) & \tau_2 \end{pmatrix}$$

is positive definite. By choosing $\tau_2 = \tau_1\kappa/\lambda$ and assuming $\lambda\kappa > 0$ i.e. the two interconnection parameters have the same sign then it follows that

$$\frac{\mathrm{d}V^\star}{\mathrm{d}t} = -\frac{\tau_1}{\lambda}(\lambda\varphi_1(\dot{\alpha})\dot{\alpha} + \kappa\varphi_2(\dot{\beta})\dot{\beta}) \leq 0$$

and the Barbashin Krasovskii La Salle theorem will give the global asymptotic stability in the sectors $(0, \infty)$ provided $\lambda\kappa > 0$

The condition $\lambda\kappa > 0$ looks more restrictive than e.g. (13). If however we let $\gamma_i \to 0$ in (13), the denominator of (13) may be written as

$$\frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2}(\gamma_1\sigma^2 + \gamma_2\rho^2) = \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_1}\right)(\gamma_1\sigma^2 + \gamma_2\rho^2) =$$
$$= \left(1 + \frac{\gamma_1}{\gamma_2}\right)\sigma^2 + \left(1 + \frac{\gamma_2}{\gamma_1}\right)\rho^2 > (\rho + \sigma)^2$$

The last inequality is due to the fact that the function

$$f(\xi) = (1 + \xi)\sigma^2 + \left(1 + \frac{1}{\xi}\right)\rho^2 \ , \ \xi \geq 0$$

has a unique minimum at $\xi = \rho/\sigma$. We deduce from (13) that

$$\lambda\kappa > -\gamma_1\gamma_2 - (\rho - \sigma)^2$$

and if $\gamma_i \to 0$ we deduce

$$\lambda\kappa > -(\rho - \sigma)^2 \tag{19}$$

which coincides with (9): this is in fact the robust version of (13) with respect to $\gamma_i > 0$. If we want this robustness extended to the natural frequencies ρ and σ, we have to consider also the case when these frequencies may be quite close one to the other , even equal. In this case $(\rho - \sigma) \to 0$ hence

$$\lambda\kappa > 0 \tag{20}$$

This shows a positive answer to the problem of Aizerman: *if robustness is taken into account the Routh Hurwitz and absolute stability conditions coincide*.

4. THE POPOV FREQUENCY DOMAIN INEQUALITY

It has been established when the positiveness theory was constructed, based on Yakubovich Kalman Popov lemma, that the most general Liapunov function of the form “quadratic form plus integral of the nonlinearity” is given by the Popov frequency domain inequality *via* Yakubovich Kalman Popov lemma. Since (18) is only quadratic, we may hope to find improved stability conditions from another Liapunov function.

A. In order to apply the frequency domain method, we take $\mu_i(t) = -\varphi_i(\nu_i(t))$ in (6) to obtain the linear subsystem

$$\begin{aligned} \ddot{\alpha} + \rho^2\alpha + \kappa\dot{\beta} &= \mu_1(t) \\ \ddot{\beta} + \sigma^2\beta - \lambda\dot{\alpha} &= \mu_2(t) \\ \nu_1 = \dot{\alpha}\ ,\ \nu_2 &= \dot{\beta} \end{aligned} \tag{21}$$

which is a 2×2 linear block. Its matrix transfer function is

$$H(s) = \frac{1}{(s^2+\rho^2)(s^2+\sigma^2)+\lambda\kappa s^2}\times \\ \times\begin{pmatrix} s^2+\sigma^2 & -\kappa s \\ \lambda s & s^2+\rho^2 \end{pmatrix} \tag{22}$$

The multivariable Popov inequality is matrix like

$$\Re e(P+\imath\omega Q)H(\imath\omega) = \frac{1}{2}(P+\imath\omega Q)H(\imath\omega)+ \\ +\frac{1}{2}H^*(-\imath\omega)(P-\imath\omega Q)\geq 0 \tag{23}$$

where P and Q are diagonal matrices with nonnegative entries

$$P = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}\ ,\ Q = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}\ ,\ \tau_i\geq 0\ ,\ \theta_i \geq 0 \tag{24}$$

and the inequality (23) is in the sense of the quadratic forms. In detail (23) reads

$$\frac{1}{(\rho^2-\omega^2)(\sigma^2-\omega^2)-\lambda\kappa\omega^2}\times \\ \times\begin{pmatrix} -\omega^2\theta_1(\sigma^2-\omega^2) & \frac{1}{2}\kappa\omega^2(\tau_1+\imath\omega\theta_1)- -\frac{1}{2}\lambda\omega^2(\tau_2-\imath\omega\theta_2) \\ \frac{1}{2}\kappa\omega^2(\tau_1-\imath\omega\theta_1)- -\frac{1}{2}\lambda\omega^2(\tau_2+\imath\omega\theta_2) & -\omega^2\theta_2(\rho^2-\omega^2) \end{pmatrix}\geq 0$$

The first Sylvester condition is

$$\frac{\theta_1\omega^2(\omega^2-\sigma^2)}{(\rho^2-\omega^2)(\sigma^2-\omega^2)-\lambda\kappa\omega^2}\geq 0\ ,\ \forall\omega\geq 0 \tag{25}$$

and since the denominator has two pairs of imaginary roots, the changes of sign are unavoidable unless $\theta_1 = 0$. The Sylvester determinant condition requires $\theta_2 = 0$ and is in this case

$$0 > \frac{1}{4}\left|\frac{(\kappa\tau_1-\lambda\tau_2)\omega^2}{(\rho^2-\omega^2)(\sigma^2-\omega^2)-\lambda\kappa\omega^2}\right|^2, \tag{26}$$

the only choice being $\tau_1\kappa = \tau_2\lambda$. The result is thus *an identically zero frequency domain condition* i.e. a limit case of the theory of positiveness Popov (1973).

B. We shall examine in the sequel the application of the positiveness theory in a limit multivariable case. The state equations for (21) are as follows

$$\begin{aligned} \dot{\alpha} &= \omega_\alpha \\ \dot{\omega_\alpha} &= -\rho^2\alpha - \kappa\omega_\beta + \mu_1(t) \\ \dot{\beta} &= \omega_\beta \\ \dot{\omega_\beta} &= \lambda\omega_\alpha - \sigma^2\beta + \mu_2(t) \end{aligned} \tag{27}$$

$$\nu_1 = \omega_\beta\ ,\ \nu_2 = \omega_\beta \tag{28}$$

hence

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\rho^2 & 0 & 0 & -\kappa \\ 0 & 0 & 0 & 1 \\ 0 & \lambda & -\sigma^2 & 0 \end{pmatrix}\ ;\ B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{29}$$

We have $\det(B\ AB) = -1 \neq 0$ hence

$$\operatorname{rank}(B\ AB\ A^2B\ A^3B) = 4$$

the pair being thus controllable. The integral index of the problem will be

$$\chi(0,t) = \tau_1\int_0^t \mu_1(\vartheta)\nu_1(\vartheta)\mathrm{d}\vartheta + \tau_2\int_0^t \mu_2(\vartheta)\nu_2(\vartheta)\mathrm{d}\vartheta \tag{30}$$

since we already known that the only choice for $\theta_i \geq 0$ is $\theta_1 = \theta_2 = 0$. Therefore

$$\mathcal{F}(u,x) = \tau_1\mu_1 c_1^* x + \tau_2\mu_2 c_2^* x \tag{31}$$

where we denoted

$$L^* = \frac{1}{2}\begin{pmatrix} 0 & \tau_1 & 0 & 0 \\ 0 & 0 & 0 & \tau_2 \end{pmatrix} \tag{32}$$

A direct computation of the characteristic function Popov (1973) will give

$$H(z,s) = \\ = \frac{1}{2}\frac{s}{(s^2+\rho^2)(s^2+\sigma^2)+\lambda\kappa s^2}\begin{pmatrix} \tau_1(s^2+\sigma^2) & -\tau_1\kappa s \\ \tau_2\lambda s & \tau_2(s^2+\rho^2) \end{pmatrix} \\ +\frac{1}{2}\frac{z}{(z^2+\rho^2)(z^2+\sigma^2)+\lambda\kappa z^2}\begin{pmatrix} \tau_1(z^2+\sigma^2) & \tau_2\lambda z \\ -\tau_1\kappa z & \tau_2(z^2+\rho^2) \end{pmatrix}$$

If we keep in mind the necessary choice $\tau_1\kappa = \tau_1\lambda$ we find immediately that

$$H(-s,s)\equiv 0\ ,\ \pi(-s,s)\equiv 0\ ,\ H(-\imath\omega,\imath\omega)\equiv 0 \tag{33}$$

where, according to the notations of Popov (1973), we have

$$\pi(z,s) = \det(sI-A)\cdot\det(zI-A^*)H(z,s) \tag{34}$$

Turning now to the multivariable equations of positiveness

$$\begin{aligned} K &= V^*V \\ L+NB &= WV \\ M+NA+A^*N &= WW^* \end{aligned} \tag{35}$$

we obtain immediately, since $K = 0$ in (31), that $V = 0$. We have also $M = 0$ in (31) and, since $\pi(-s,s)\equiv 0$, the positiveness theorem of Popov (1973) will give $W = 0$; equations (35) become

$$L+NB = 0\ ,\ NA+A^*N = 0 \tag{36}$$

where $N \leq 0$. The equations are solved directly to obtain a diagonal matrix

$$N = -\frac{1}{2}\mathrm{diag}\left\{\tau_1\rho^2\ ,\ \tau_1\ ,\ \tau_2\sigma^2\ ,\ \tau_2\right\}\ ,\ \tau_1\kappa = \tau_2\lambda \quad (37)$$

The prescribed Liapunov function is quadratic and has the form

$$V(x) = -x^*Nx = \frac{1}{2}\tau_1\left[\rho^2\alpha^2 + \omega_\alpha^2 + \frac{\kappa}{\lambda}(\sigma^2\beta^2 + \omega_\beta^2)\right] \quad (38)$$

and is positive definite provided $\lambda\kappa > 0$ as required by the choice $\tau_i > 0$, $\tau_1\kappa = \tau_2\lambda$.

It is clear that (38) is nothing more but (18) with the same choice $\tau_1\kappa = \tau_2\lambda$. Therefore *the energy quadratic Liapunov function gives the best achievable result from the point of view of the absolute stability and of the Aizerman problem.*

5. CONCLUSION AND FUTURE RESEARCH

The last sentence of the previous section might be a valuable concluding remark for this research since we applied the entire set of Liapunov like methods to obtain the largest domains of linear and absolute stability.

However, if we remind the fact that the approach of Bulgakov (1942, 1943, 1946, 1954) was that of the self sustained oscillations, their absence of presence being considered as the absolute stability criterion, the benchmark character of the application would not be completed without this analysis. The oscillatory behavior will thus be considered in a future companion paper.

REFERENCES

M. A. Aizerman. On convergence of the control process under large deviations of the initial conditions (in russian). *Avtom. i telemekh.*, 7(2-3):148–167, June 1946.

M. A. Aizerman. On a problem on the stability of dynamical systems in the large (in russian). *Usp. Mat. Nauk*, 4(4):187–188, August 1949.

N. E. Barabanov. On the problem of kalman (in russian). *Sib. Mat. J.*, 29(3):3–11, June 1988.

B. V. Bulgakov. Self sustained oscillations of the controlled systems (in russian). *Dokl. Akad. Nauk USSR*, 37(9):283–287, September 1942.

B. V. Bulgakov. Self sustained oscillations of the controlled systems (in russian). *Prikl. Mat. Mekh.*, 7(2):97–108, April 1943.

B. V. Bulgakov. Some control theory problems for the case of the nonlinear characteristics (in russian). *Prikl. Mat. Mekh.*, 10(3):313–332, June 1946.

B. V. Bulgakov. *Oscillations (in Russian)*. Gostekhizdat, Moscow, 1st edition, 1954.

A. M. Letov. *Stability of Nonlinear Control Systems (Russian original edition 1955, second revised 1962)*. Princeton Univ. Press, Princeton NJ, 1961.

A. I. Lurie and V. N. Postnikov. About the stability of the controlled systems (in russian). *Prikl. Mat. Mekh.*, 8(3):246–248, June 1944.

V. A. Pliss. *Some problems concerning the theory of stability in the large (in Russian)*. Leningrad Univ. Publ. House, Leningrad, 1st edition, 1958.

V. M. Popov. *Hyperstability of Control Systems*. Editura Academiei & Springer-Verlag, Bucharest & Berlin-Heidelberg-New-York, 1973.

A. A. Voronov. *Stability, controllability, observability (in Russian)*. Nauka Publ. House, Moscow, 1st edition, 1979.