Bond Graph Modelling and Nonlinear Control of an Inverted Pendulum

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Abstract: The paper presents the Bond Graph modelling and the feedback linearizing technique applied to an inverted pendulum system. First, the Bond Graph model of the inverted pendulum system is achieved. Next, by using the feedback linearizing technique, a nonlinear control law is obtained. This nonlinear control method provides an alternative solution to existing classical linear methods. For the implementation of the nonlinear control law we supposed that all states are measurable. Some numerical simulation results for the controlled system are also presented.

Keywords: Bond Graph, Modelling, Feedback linearization, Inverted pendulum.

1. INTRODUCTION

The study of system dynamics resides in modelling its behaviour. Systems models are simplified, abstracted structures used to predict the behaviour of the studied systems. Our interest is pointing towards the obtaining of mathematical model used to predict certain aspects of the system response to the inputs. In mathematical notations, a system model is described by a set of ordinary differential equations in terms of state variables and a set of algebraic equations that relate the state variable to other system variables (Karnopp and Rosenberg (1974)).

In order to model a system, it is usually necessary to decompose the system into smaller parts - subsystems - that can be modelled separately. The subsystem is a part of a system that can be modelled as the system itself obtaining submodels. The overall model can then be built up by combining the separate submodels. System models can be constructed using a uniform notation for all types of physical systems, which is the Bond Graph method based on energy and information flow (Karnopp and Rosenberg (1974), Thoma (1975), Dauphin-Tanguy (2000)). Using this method it is possible to develop models of electrical, mechanical (Pastravanu and Ibanescu (2001), Damic and Montgomery (2002), Gawthrop and Bevan (2007)), magnetic, hydraulic, pneumatic, thermal, and other systems using a small set of variables.

The method uses the effort-flow analogy to describe physical processes. A Bond Graph consists of subsystems linked together by lines representing power bonds. Each process is described by a pair of variables, effort (e) and flow (f), and their product is the power. The direction of power is depicted by a half arrow.

In a dynamic system the effort and the flow variables, and hence the power fluctuate in time. It is remarkable how models of various systems belonging to different engineering domains can be express using a set of only nine elements, called elementary components. These elements are sufficient to describe any physical system regardless of the energy types processed by it (Karnopp and Rosenberg (1974)).

A classification of Bond Graph elements can be made up by the number of ports. The ports are places where interactions with other processes take place. There are one port elements represented by inertial elements (I), capacitive elements (C), resistive elements (R), effort sources (SE) and flow sources (SF), two ports elements represented by transformer elements (TF) and gyrator elements (GY), and multi ports elements - effort junctions (J0) and flow junctions (J1).

The concept of causality is an important concept embedded in Bond Graph theory. This refers to cause (input) and effect (output) relationship. Thus, as part of the Bond Graph modelling process, a causality assignment is implicitly introduced and it is graphically represented by a short stroke, called causal stroke, placed perpendicular to the bond at one of its ends indicating the direction of the effort variable. Causal stroke assignment is independent of the power flow direction. This leads to the description of Bond Graphs in the form of state – space equations.

Besides the power variables, two other types of variables are very important in describing dynamic systems and these variables, sometimes called energy variables, are the generalized momentum (p) as time integral of effort and the generalized displacement (q) as time integral of flow (Karnopp and Rosenberg (1974), Thoma (1975)).

A largely used method for the control of nonlinear systems is to calculate a linear controller for the linear approximation of the nonlinear system around an operating point. This kind of control works in a small neighbourhood of the operating point, and when the system is far from this point, the linear controller will not have the desired behaviour. Another strategy, the feedback linearization (Isidori (1995)) is a good technique because the nonlinear system is transformed into a linear system and only then a linearizing controller is applied. Therefore, a controller designed by using feedback linearization is working in any point, not only in a small neighbourhood of the operating point.

In this paper, the Bond Graph modelling method is applied to the Quanser Inverted Pendulum experiment. Also, the method of nonlinear control and state feedback linearization is applied to this system in order to achieve a control law. The obtained nonlinear law, together with our nonlinear system, achieves input-output linearization (Bobasu et al. (1998a,b), Bodson et al. (1994), Fossard and Normand-Cyrot (1993), Raumer et al. (1994)). By using the feedback linearizing techniques, it is established a nonlinear control law so that the input-output behaviour of closed loop system is the same as those of a linear system.

2. BOND GRAPH MODEL OF THE INVERTED PENDULUM

The Inverted Pendulum experiment consisting of a rotary plant module and a pendulum arm is depicted in Fig. 1.

Fig. 1. Quanser Inverted pendulum system

This experiment consists of a SRV02 rotary plant module and a pendulum module. The SRV02 rotary plant module serves as the base component for the rotary family of experiments. Its modularity facilitates the change from one experimental setup to another and it consists of a DC motor in a solid aluminium frame equipped with a gearbox whose output drives external gears.

The pendulum module is attached to the SRV02 load gear by two thumbscrews. The Inverted Pendulum experiment is a classical example of how the use of control may be employed to stabilize an inherently unstable system. It is also an accurate model in the pitch and yaw of a rocket in flight and can be used as a benchmark for many control methodologies.

The goal of this section is to model the Quanser Rotary Inverted Pendulum experiment using a systematic way of building it in small steps (Damic and Montgomery (2002)).

A first step is to write a word Bond Graph which contains words instead of standard symbols for the main components, and bonds for power and signal exchange.

The next step is to replace words by standards elements which contain precise mathematical or functional relations. When the Bond Graph model is done, it is possible to formulate the state space equations starting from the constitutive relations of elements.

We proceed to the design of the Bond Graph model by identifying the system components and connecting them as they are in the real system (see Damic and Montgomery (2002)):

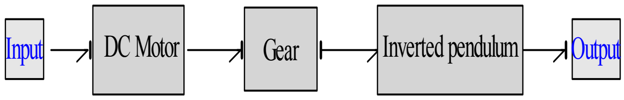

Fig. 2. Word Bond Graph of the system

The gyrator GY from the DC Motor component describes the electromechanical conversion in the motor relating the back electromechanical force (emf) from the electrical part to the angular velocity of the rotor from the mechanical part, respectively the armature current from the electrical part to the torque acting on the rotor. For this reason, the gyrators are called overcrossed transformers.

$$\begin{cases} V_b = k_m \cdot \omega \\ T_m = k_t \cdot i_a \end{cases} \tag{1}$$

where: k_t is the motor torque constant, and k_m is the back emf constant.

The electrical process in the armature is described in Bond Graph terms by the armature resistance R_a represented using a resistive element (R), and the armature inductance L_a represented using an inertial element (I). These two elements are joined through an 1 junction. The mechanical process is also described using an inertial element that models the rotation of the rotor mass moment of inertia J_m, and a resistive element that models the linear friction coefficient B_m. These two elements are joined through an 1 junction. The gearbox named Gear is represented by a transformer (TF) element having its parameter equal to the reduction ratio of the gearbox, an inertial element, representing the equivalent high gear inertia J_g and a resistive element to model the viscous friction forces.

In order to model the component Pendulum it is required to write the equations of motion of inverted pendulum. Fig. 3 depicts the pendulum in motion, where α is the pendulum arm deflection and θ the servo load gear angle

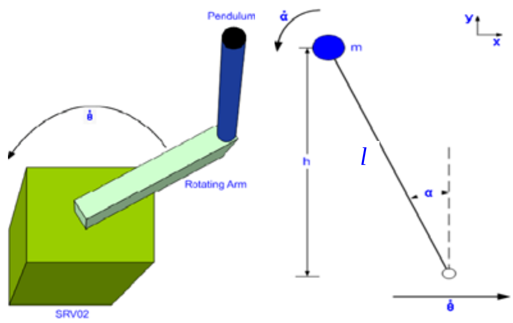

Fig. 3. Top and side of the inverted pendulum in motion

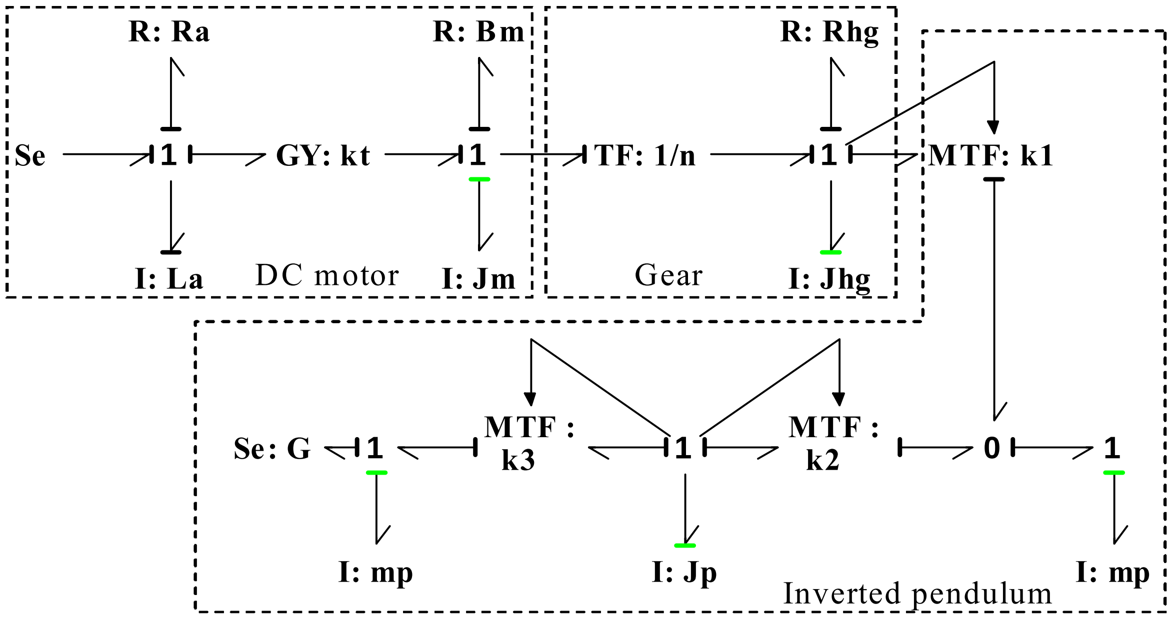

Fig. 4. Bond Graph model of the Inverted Pendulum

We shall begin the derivation by examining the velocity of the pendulum centre of mass. As it can be seen in Fig. 3, there are two components for the velocity of the pendulum lumped mass:

$$v_{pcm} = -l\cos\alpha(\dot\alpha)\hat{x} - l\sin\alpha(\dot\alpha)\hat{y} \tag{2}$$

We also know that the pendulum arm is moving with the rotating arm at a rate of:

$$v_{arm} = r\dot\theta \tag{3}$$

Using (2) and (3) the velocities for x-direction and y-direction are given by (4):

$$\begin{aligned} v_x &= r\dot\theta - l\cos\alpha(\dot\alpha) \\ v_y &= -l\sin\alpha(\dot\alpha) \end{aligned} \tag{4}$$

We used the above three equations (2), (3) and (4) in order to model the dependencies between the velocities involved in the system $\dot\theta$, v_{arm}, $\dot\alpha$, $\dot x$ and $\dot y$ in the Bond Graph model presented in Fig. 4. Thus the angular velocity with respect to rotational gear and the velocity of the pendulum arm are related using a transformer element, a modulated one, with the transformer modulus $k_1 = r$ (the pendulum arm length).

Other two modulated transformers were used to relate the angular velocity of the pendulum, $\dot\alpha$, with the velocities on the x-direction, $\dot x$, and y-direction, $\dot y$. These elements are characterized by $k_2 = -l\cos(\alpha)$ and respectively $k_3 = -l\sin(\alpha)$.

The Bond Graph model of the Pendulum component also contains an inertial element that models the moment of inertia of the pendulum and two inertial elements to model the pendulum mass over the x and y-directions. The weight force G was modelled using an effort source.

3. ANALYTICAL MODEL OF INVERTED PENDULUM

The dynamical equations of the system can be also obtained by using the Euler-Lagrange formulation, taking into account the relations of the velocities of pendulum. The potential and kinetic energy are calculated considering the gravity for the potential energy and the total kinetic energy of the pendulum as the energy of the point mass plus the kinetic energy of the pendulum rotating about its center of mass.

The system is characterized by a set of two differential equations:

$$\begin{aligned} &\left(mr^2 + J\right)\ddot\theta + mrl\ddot\alpha\cos\alpha - mrl\dot\alpha^2\sin\alpha = T \\ &mrl\ddot\theta\cos\alpha - mrl\dot\alpha\dot\theta\sin\alpha + ml^2\ddot\alpha - gml\sin\alpha = 0 \end{aligned}$$

where l represents length to pendulum's center mass, m is the mass of pendulum arm, r is the rotating arm length, θ the servo load gear angle (radians), α is the pendulum arm deflection (radians), J is the pendulum inertia about its center of mass and g gravitational acceleration.

The state space representation has the following form:

$$\dot x = f(x) + g(x)u, y = h(x) \tag{5}$$

where we choose as state variables

$$x^T = [x_1, x_2, x_3, x_4] = \left[\theta(t), \alpha(t), \dot\theta(t), \dot\alpha(t)\right]$$

For f and g we obtained the following relations:

$$f(x) = \begin{bmatrix} x_3 \\ x_4 \\ -B\sin x_2\left(\dfrac{Cx_4^2}{E(x)} - \cos x_2\left(Bx_3x_4 + D\right)\right) \\ \dfrac{B^2x_4^2\cos x_2 - A\left(Bx_3x_4 + D\right)}{E(x)}\sin x_2 \end{bmatrix}, \tag{6}$$

$$g(x)=\begin{bmatrix}0\\0\\-\dfrac{C}{E(x)}\\\dfrac{B\cos x_2}{E(x)}\end{bmatrix}, \tag{7}$$

where we denoted:

$E(x)=B^2\cos^2 x_2 - AC$,

$A=mr^2+J$, $B=mrl$,

$C=ml^2$, $D=mgl$.

As the output variable, the angular position $\alpha(t)$ is considered:

$$y=h(x)=\alpha(t)=x_2(t) \tag{8}$$

4. THE FEEDBACK LINEARIZING METHOD

We consider the following form of the state space equations in the case of nonlinear system:

$$\begin{aligned}\dot{x} &= f(x)+\sum_{i=1}^{m} g_i(x)u_i\\ y_j &= h_j(x) \qquad j=1...m\end{aligned} \tag{9}$$

where $f(x), g_1(x), g_2(x),..., g_m(x)$ are smooth vector fields (see Isidori (1995), Fossard and Normand-Cyrot (1993)).

The problem of exact linearization via feedback and diffeomorphism consists in transforming a nonlinear system (9) into a linear one using a state feedback and a coordinate transformation of the systems state.

Let's introduce now the Lie derivative of the function $h(x): R^n \to R$ along the vector field $f(x)=[f_1(x),..., f_n(x)]$:

$$L_f h(x)=\sum_{i=1}^{n}\frac{\partial h(x)}{\partial x_i} f_i(x) \tag{10}$$

Definition. A multivariable nonlinear system of the form (9) has a relative degree $\{r_1,..., r_m\}$ at a point x^0 if:

$$L_{g_j} L_f^k h_i(x)=0 \tag{11}$$

for all $1\le j\le m$, for all $1\le i\le m$, for all $k\le r_i-1$, and for x in a neighbourhood of x^0, the $m\times m$ matrix:

$$A(x)=\begin{bmatrix} L_{g_1}L_f^{r_1-1}h_1(x) & . & . & L_{g_m}L_f^{r_1-1}h_1(x)\\ L_{g_1}L_f^{r_2-1}h_2(x) & . & . & L_{g_m}L_f^{r_2-1}h_2(x)\\ . & . & . & .\\ L_{g_1}L_f^{r_m-1}h_m(x) & . & . & L_{g_m}L_f^{r_m-1}h_m(x)\end{bmatrix} \tag{12}$$

is nonsingular at $x=x^0$.

Remark: Let be a SISO nonlinear system of the form (9), which has the relative degree r at a point x^0.

The state feedback:

$$u=\frac{1}{L_g L_f^{r-1}h(x)}\left[-L_f^r h(x)+v\right]$$

transforms the nonlinear system into a system, whose input-output behaviour is the same with a linear system having the transfer function:

$$H(s)=\frac{1}{s^r}.$$

Theorem. Let be the nonlinear system of the form (9). Suppose the matrix $g(x^0)$ has rank m. Then the state space exact linearization problem is solvable if and only if: for each $0\le i\le n-1$, the distribution G_i has constant dimension near x^0; the distribution G_{n-1} has dimension n; for each $0\le i\le n-2$, the distribution G_i is involutive.

For system (5) we have:

$$\begin{aligned}&L_f^0 h(x)=0;\, L_g L_f^0 h(x)=0;\, L_g L_f^1 h=\frac{B\cos x_2}{E(x)}\\ &L_f^2 h(x)=\frac{B^2 x_4^2\cos x_2 - A(Bx_3x_4+D)}{E(x)}\sin x_2\end{aligned} \tag{13}$$

Thus, we see that the system has relative degree $r=2$. In this situation, the state feedback:

$$u=\frac{1}{L_g L_f^1 h(x)}(-L_f^2 h(x)+v) \tag{14}$$

transforms the system (5) into a system whose input-output behavior is identical to that a linear system having a transfer function: $H(s)=\dfrac{1}{s^2}$.

Imposing on the linear system an additional feedback of the form:

$$v=c_0(\alpha_{ref}-x_2)-c_1\dot{x}_2 \tag{15}$$

then, the obtained system has a linear input-output behavior, described by the following transfer function

$$H(s)=\frac{c_0}{s^2+c_1 s+c_0} \tag{16}$$

In relation (15) α_{ref} is the imposed reference of the arm deflection.

The coefficients c_0, c_1 in (16) are determined using a pole placement procedure. The values of these coefficients are chosen such that the behaviour of the entire closed loop system has a desired shape (Bobasu et al. (1998a)).

5. SIMULATION RESULTS

In order to test the behaviour and the performance of the proposed nonlinear control strategy, extensive simulations were performed using the Quanser Inverted Pendulum experiment.

The values of the inverted pendulum parameters are:

$m = 0.128kg; r = 0.158m; l = 0.35\,m;$

$\mathrm{J} = 0.0015\,\mathrm{kgm}^2; \mathrm{g} = 9.81\,\mathrm{m/s}^2$

The design parameters are set to:

$c_0 = 100$, $c_1 = 14$

Fig. 5 presents the time evolution of the pendulum arm deflection α (in radians) for $\alpha_{ref} = 0$.

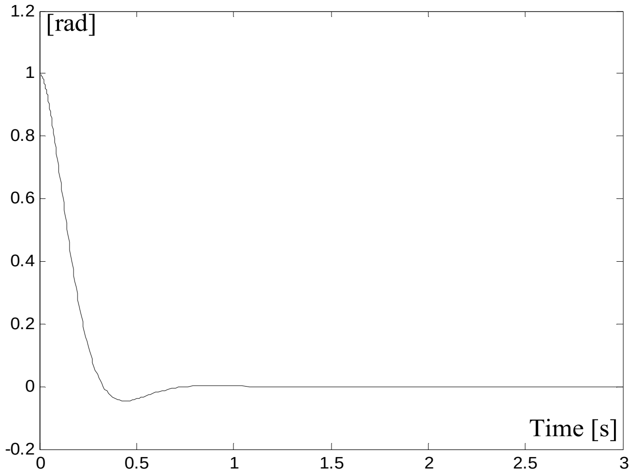

Fig. 5. Time evolution of pendulum arm deflection

It can be seen that the behaviour of the controlled system is quite good, and the inverted pendulum is stabilized.

In Fig. 6 the time profile of the angular velocity $\dot{\alpha}$ is shown.

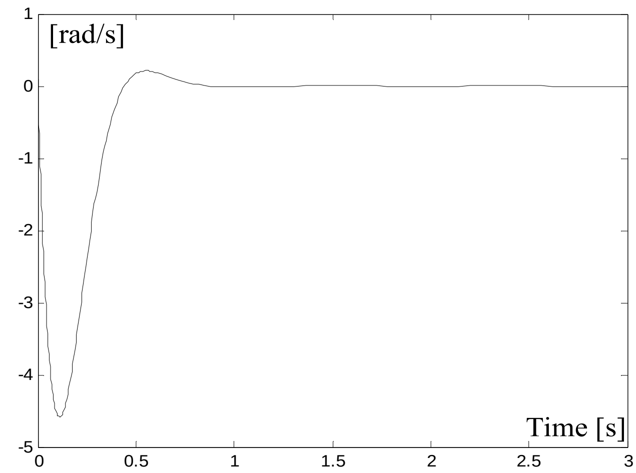

Fig. 6. Time evolution of pendulum angular velocity $\dot{\alpha}$

6. CONCLUSIONS

In this paper our interest was pointed towards the Bond Graph modelling and the design of feedback linearizing technique for an Inverted Pendulum system.

The Bond Graph model of the system was built up writing first the word Bond Graph containing words instead of standard symbols for the main components and bonds for power and signal exchange, and then replacing words by standards elements which contain precise mathematical or functional relations. The system was decomposed into three subsystems that were modelled separately. By joining together these three models, we obtained the complete Bond Graph model of the Quanser Rotary Inverted Pendulum system. The model was created and simulated using 20sim modelling and simulation environment.

The nonlinear control method based on the feedback linearizing technique provides an alternative solution to existing classical linear methods. The implementation of the method requires a complete knowledge of the state variables (or the use of a state observer).

REFERENCES

Karnopp, D., and Rosenberg, R. (1974). *System Dynamics: A Unified Approach*. John Wiley, New York.

Thoma, J. (1975). *Introduction to Bond Graphs and Their Applications.* Perg. Press, Oxford.

Dauphin-Tanguy, G. (2000). *Les Bond Graphs.* Hermes Sci., Paris.

Păstrăvanu, O., Ibănescu, R. (2001). *Bond-graph language in modeling and simulation of physical-technical systems*. Gh. Asachi, Iasi.

Damic, V., Montgomery, J. (2002). *Mechatronics by Bond Graphs*. Springer, Germany.

Gawthrop, P.J., and Bevan, G.P. (2007). A Tutorial Introduction for Control Engineers. *IEEE Control Systems*, 27, 24-45.

Isidori, A. (1995). *Nonlinear control systems*. Springer-Verlag, Berlin.

Bobasu, E., Petre, E., Popescu, D. (1998). On nonlinear control for electric induction motors. In *Process Control'98*, (1), 44-47. Pardubice, Czech Republic.

Bobasu, E., Ionete, C., Selisteanu, D. (1998). On nonlinear control for electric d.c. motors. In *Process Control'98*, (1), 42-46. Pardubice, Slovak Republic.

Bodson, M., Chiasson, J., Novotnac R. (1994). High-Performance Induction Motor Control Via Input-Output Linearization. *IEEE Contr. Systems*, 14 (4), 25-33.

Fossard, A.J., Normand-Cyrot, D. (1993). *Systemes nonlineaires*. Masson, Paris.

Raumer, T., Dion, J. M., Dugard, L., Thomas, J.L. (1994). Applied Nonlinear Control of an Induction Motor Using Digital Signal Processing. *IEEE Trans. Contr. Systems Technology*, 2 (4), 327- 335.